SOME INDEX FORMULÆ ON THE MODULI SPACE OF STABLE PARABOLIC VECTOR BUNDLES

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ABSTRACT. We study natural families of $\overline{\partial}$-operators on the moduli space of stable parabolic vector bundles. Applying a families index theorem for hyperbolic cusp operators from our previous work, we find formulæ for the Chern characters of the associated index bundles. The contributions from the cusps are explicitly expressed in terms of the Chern characters of natural vector bundles related to the parabolic structure. We show that our result implies formulæ for the Chern classes of the associated determinant bundles consistent with a recent result of Takhtajan and Zograf.

INTRODUCTION

The Atiyah-Singer index theorem has various generalizations on non-compact manifolds and manifolds with boundary, among the most famous being the Atiyah-Bott index theorem [5] for local elliptic boundary conditions and the Atiyah-Patodi-Singer index theorem [4] for global elliptic boundary conditions or cylindrical ends. Part of the reason explaning the great variety of possible generalizations is that, on a non-compact manifold, the index of an elliptic operator depends in a subtle way on its behavior at infinity. For instance, in contrast to closed manifolds, elliptic operators are not necessarily Fredholm. Some extra conditions have to be satisfied at infinity, the precise conditions depending on which type of operators one considers.

In the school of Melrose, the non-compact manifold is described as the interior of a manifold with boundary (or a manifold with corners) so that the behavior of the operator at infinity is encoded by a symbol defined on the boundary – usually called the normal operator or the indicial family. A corresponding calculus of pseudodifferential operators in which one can construct parametrices then allows one to identify the elliptic operators which are Fredholm as those whose normal operator is invertible, intuitively an ellipticity condition at infinity. Over the years, various types of pseudodifferential calculi have been introduced on non-compact manifolds , for instance the b-calculus [29], [27], the Θ-calculus [17], the 0-caculus [22], the edge calculus [21], the scattering calculus [28] or the fibred cusp calculus [23]. These

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calculi are usually associated to the asymptotic behavior at infinity of some complete Riemannian metric in the sense that the associated Laplacian or Dirac type operator is an element of the calculus. More generally, one can talk about a Lie structure at infinity [25] and a general procedure to get a corresponding pseudodifferential calculus has been obtained by Ammann, Lauter and Nistor [3] using groupoids.

We are interested in the situation where the non-compact manifold considered is a Riemann surface $\Sigma = \overline{\Sigma} \setminus \{p_1, \ldots, p_n\}$ of genus g with n punctures. Provided $2g - 2 + n > 0$, such a Riemann surface admits a canonical hyperbolic metric g_Σ with conformal class prescribed by the complex structure. Near infinity, that is, near a puncture, the geometry associated to such a metric is the one of a cusp. The pseudodifferential operators one gets out of this geometry at infinity are called hc-operators (or d-operators in the terminology of Vaillant [39]). In his thesis [39], Vaillant studied in great detail the Dirac type operators associated to geometries that asymptotically behave like a cusp or a fibred cusp on a non-compact manifold (not necessarily a punctured Riemann surface). He gave an explicit description of the continuous spectrum in terms of an operator at infinity and, in the Fredholm case, provided an index formula involving the usual Atiyah-Singer term in the interior and some eta forms defined in terms of the normal operator.

In general, eta forms are very hard to compute. However, on punctured Riemann surfaces, the geometry at infinity is sufficiently simple to allow explicit computations. In [2], using the generalization of Vaillant's index theorem to families from [1], this fact was put to use to get a local index theorem in terms of the Mumford-Morita-Miller classes for families of $\overline{\partial}$-operators parametrized by the moduli space of Riemann surfaces of genus g with n marked points. Using heat kernel techniques as in [9] (see also §9-10 in [6]), it was also possible to give an alternate proof of the formula of Takhtajan and Zograf [37] for the curvature of the Quillen connection defined on the corresponding determinant line bundle.

In this paper, we want to apply the generalization of Vaillant's index theorem to families (Theorem 4.5 in [1]) to another moduli space, namely, the moduli space of stable parabolic vector bundles with vanishing parabolic degree on a Riemann surface $\overline{\Sigma}$ with marked points. A parabolic vector bundle is a holomorphic vector bundle on $\overline{\Sigma}$ together with a parabolic structure specified at each marked point by a set of weights and multiplicities. According to [12], for generic weight systems, the moduli space $\mathcal{N}$ of stable parabolic vector bundles of vanishing parabolic degree admits a universal parabolic vector bundle, that is, a holomorphic vector bundle $E \to \overline{\Sigma} \times \mathcal{N}$ such that for each $m \in \mathcal{N}$, $E|_{\overline{\Sigma}\times\{m\}}$ represents the parabolic vector bundle described by the point m. In this context, one gets a family of $\overline{\partial}$-operators $\overline{\partial}_E$ parametrized by $\mathcal{N}$ acting fibrewise on sections of $E|_{\overline{\Sigma}\times\{m\}}$ above $m \in \mathcal{N}$. Similarly, one gets a family of $\overline{\partial}$-operators for the endomorphism bundle $\mathcal{E} := \mathrm{End}(E)$. This family is of particular importance for the moduli space

$\mathcal{N}$ since there is a canonical identification of the tangent bundle of $\mathcal{N}$ with the cokernel bundle coker $\overline{\partial}_{\mathcal{E}} \to \mathcal{N}$.

In [38], Takhtajan and Zograf were able to define the Quillen connection on the determinant line bundle of the family $\overline{\partial}_{\mathcal{E}}$ using an appropriate Selberg Zeta function. More importantly, they obtained an explicit formula for its curvature, and consequently for the first Chern form of the tangent bundle of $\mathcal{N}$. Their formula involves the usual Atiyah-Singer term and a cuspidal defect.

In this paper we show that, as operators on a punctured Riemann surface Σ, $\overline{\partial}_E$ and $\overline{\partial}_{\mathcal{E}}$ are smooth families of Fredholm Dirac type hc-operators, so that the families index formula of [1] applies to them. We then perform a computation of the eta forms to express them in terms of explicit data coming from the parabolic structure. This leads to local families index formulæ for the families $\overline{\partial}_E$ and $\overline{\partial}_{\mathcal{E}}$ (theorem 4.2 and theorem 4.4). In both cases, we are also able, as in [2], to define the Quillen metric on the determinant line bundle via heat kernel techniques and identify its curvature with the 2-form part of the index formula (theorem 5.1 and theorem 5.4). For the determinant line of the family $\overline{\partial}_{\mathcal{E}}$, our formula agrees with the one of Takhtajan and Zograf (Theorem 2 in [38]).

The paper is organized as follows. In §1, we review the definitions and the various properties surrounding the notion of parabolic vector bundle. In §2, we describe how the $\overline{\partial}$-operator associated to a parabolic vector bundle can be seen as a Dirac type hc-operator and we use the criterion of Vaillant to check that it is Fredholm. In §3, we consider families of such operators parametrized by the moduli space of stable parabolic vector bundles of parabolic degree equal to zero. In particular, following [38], we describe the natural connection that can be put on the universal parabolic vector bundle $E \to \Sigma \times \mathcal{N}$. In §4, we compute explicitly the eta forms involved in the index formulæ and get our main results, theorem 4.2 and theorem 4.4. Finally, in § 5, we define the Quillen connection of the determinant line bundle of the families $\overline{\partial}_E, \overline{\partial}_{\mathrm{End}(E)}$ and compute their curvature. In Appendix A, we also quickly indicate how the short time asymptotic of the regularized trace of the heat kernel can be deduced from the work of Vaillant [39] and the pushforward theorem.

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1. Stable parabolic vector bundles

Let $\overline{\Sigma}$ be a compact Riemann surface of genus g and negative Euler characteristic (i.e., $g \geq 2$). It is well-known that a representation of the fundamental group

$$\rho : \pi_1(\overline{\Sigma}) \to \mathrm{U}(k)$$

gives rise to a holomorphic vector bundle over $\overline{\Sigma}$, E_ρ. Indeed, since $\mathbb{H}$ is the universal covering space of $\overline{\Sigma}$, it suffices to take the product $\mathbb{H}\times\mathbb{C}^k$ and mod out by the action of $\pi_q(\overline{\Sigma})$ induced by ρ. Furthermore, the trivial metric and connection on the bundle $\mathbb{H}\times\mathbb{C}^k\to\mathbb{H}$ descend to a Hermitian metric and compatible flat holomorphic connection on E_ρ. This connection determines ρ as its holonomy representation; which shows that only vector bundles arising from complex representations of the fundamental group admit flat holomorphic connections.

It is also possible to give a criterion in terms of geometric invariant theory to describe what kind of holomorphic vector bundles arises in this way. If E is a holomorphic vector bundle over $\overline{\Sigma}$, its slope is the quotient

$$\mu(E) := \frac{\deg(E)}{\operatorname{rank}(E)} \tag{1.1}$$

where $\deg(E) = c_1(E)[\overline{\Sigma}]$ is the degree of E. The vector bundle E is said to be stable if whenever F is a holomorphic sub-bundle, we have

$$\mu(F) < \mu(E).$$

A theorem of Narasimhan and Seshadri [32] asserts that a holomorphic vector bundle E of degree zero is stable if and only if it is isomorphic to a vector bundle induced from an **irreducible** representation $\rho:\pi_1(\overline{\Sigma})\to \mathrm{U}(k)$ where k is the rank of E. In particular, a holomorphic vector bundle E is of the form E_ρ for some unitary representation ρ if and only if it is a direct sum of stable vector bundles.

Donaldson [14] gave a new proof of the theorem of Narasimhan and Seshadri using gauge theory. He found a geometric interpretation of stability by showing that a holomorphic bundle E over $\overline{\Sigma}$ is stable if and only if there is a unitary connection on E having constant central curvature $R = -2\pi i\mu(E)(*1)$. Since this implies that stable vector bundles of degree zero admit a flat connection, the theorem of Narasimhan and Seshadri follows immediatly.

Next consider the situation where the Riemann surface $\overline{\Sigma}$ has some marked points. Precisely, let $\overline{\Sigma}$ be a compact Riemann surface of genus g and $S=\{p_1,\dots,p_n\}$ a subset consisting of n distinct points. Assume that the Euler characteristic of the punctured Riemann surface $\Sigma := \overline{\Sigma}\setminus S$ (equal to $2-2g-n$) is negative. By the uniformization theorem for Riemann surfaces, one can represent the punctured Riemann surface Σ as the quotient

$$\Sigma = \Gamma\setminus\mathbb{H}, \quad \Gamma\cong\pi_1(\Sigma) \tag{1.2}$$

of the hyperbolic half-plane $\mathbb{H} := \{z\in\mathbb{C} \mid \operatorname{Im}(z)>0\}$ by a torsion-free Fuchsian group Γ generated by hyperbolic transformations $A_1, B_1, \cdots, A_g, B_g$ and parabolic transformations $S_1,\dots,S_n$ satisfying the single relation

$$A_1B_1A_1^{-1}B_1^{-1}\cdots A_gB_gA_g^{-1}B_g^{-1}S_1\cdots S_n = 1. \tag{1.3}$$

Let $x_1, \ldots, x_n$ be the fixed points of $S_1, \ldots, S_n$ and let $\overline{\mathbb{H}}$ be the union of $\mathbb{H}$ with the set of all points fixed by a parabolic element of Γ. The action of Γ naturally extends to $\overline{\mathbb{H}}$ so that $\overline{\Sigma} \cong \Gamma \setminus \overline{\mathbb{H}}$ and the image of $x_1, \ldots, x_n$ under the quotient map are precisely the marked points $p_1, \ldots, p_n$ on $\overline{\Sigma}$. For the fixed point x_i of S_i, it is convenient to choose an element $\sigma_i \in \mathrm{SL}(2, \mathbb{R})$ such that $\sigma_i(\infty) = x_i$ and

$$\sigma_i^{-1} S_i \sigma_i = \begin{pmatrix} 1 & \pm 1 \\ 0 & 1 \end{pmatrix}.$$

This provides a local coordinate $\zeta_i = e^{2\pi i \sigma_i^{-1} z}$ near $p_i \in \Gamma \setminus \overline{\mathbb{H}}$.

Let $\rho : \Gamma \to \mathrm{U}(k)$ be a unitary representation, where $\mathrm{U}(k)$ is the group of $k \times k$ unitary matrices. Using the representation ρ, we can define an action of the group Γ on the trivial vector bundle $\mathbb{H} \times \mathbb{C}^k$ by

$$\begin{array}{rccc} \gamma : & \mathbb{H} \times \mathbb{C}^k & \to & \mathbb{H} \times \mathbb{C}^k \\ & (z, v) & \mapsto & (\gamma z, \rho(\gamma) v) \end{array} \tag{1.4}$$

The quotient of this action define a flat Hermitian bundle E_ρ on the punctured Riemann surface $\Sigma := \overline{\Sigma} \setminus \{p_1, \ldots, p_n\}$. As described in ([24], definition 1.1), one can also obtain a holomorphic vector bundle on $\overline{\Sigma}$ by considering the sheaf $\mathcal{V}$ of Γ-invariant holomorphic sections of $\mathbb{H} \times \mathbb{C}^k$ which are bounded at the cusps. This is playing the rôle of the sheaf of Γ-invariant holomorphic sections of $\overline{\mathbb{H}} \times \mathbb{C}^k$. The direct image of this sheaf under the canonical map $p^\Gamma : \overline{\mathbb{H}} \to \overline{\Sigma}$ is locally free of rank k, so define a holomorphic vector bundle

$$\overline{E}_\rho \to \overline{\Sigma} \tag{1.5}$$

of rank k. This is a typical example of a parabolic vector bundle, the definition of which we now recall.

Definition 1.1. *A **parabolic structure** on a holomorphic vector bundle $\pi : E \to \overline{\Sigma}$ consists in giving at each point $p \in S$*

- *a flag $E_p = F_1 E_p \supset F_2 E_p \supset \cdots \supset F_{r(p)} E_p \supset F_{r(p)+1} E_p = \emptyset$,*
- *weights $\alpha_1(p), \cdots, \alpha_{r(p)}(p)$ associated to $F_1 E_p, \ldots, F_{r(p)} E_p$ in such a way that $0 \le \alpha_1(p) < \alpha_2(p) < \cdots < \alpha_{r(p)}(p) < 1$.*

*The **mutiplicity** of the weight $\alpha_i(p)$ is $k_i(p) := \dim F_i E_p - \dim F_{i+1} E_p$. A **parabolic vector bundle** is a holomorphic vector bundle equipped with a parabolic structure.*

For the bundle $\overline{E}_\rho$, the natural parabolic structure is specified at $p_i \in S$ by the eigenspaces and the eigenvalues of $\rho(S_i)$ where S_i is the parabolic element fixing the cusp associated to p_i. Indeed, let

$$\mathbb{C}^k = \bigoplus_{j=1}^{r(p_i)} \overline{E}^j_{p_i} \tag{1.6}$$

be a decomposition of $\mathbb{C}^k$ in terms of the eigenspaces $\overline{E}^j_{p_i}$ of $\rho(S_i)$ with eigenvalue $\lambda_j(p_i) = e^{2\pi i \alpha_j(p_i)}$. Assume that they are ordered in such a way

that $0 \leq \alpha_1(p_i) < \alpha_2(p_i) < \cdots < \alpha_{r_i}(p_i) < 1$. Then the parabolic structure at p_i is given by

$$F_j E_{p_i} := \bigoplus_{m=j}^{r_i} \overline{E}^m_{p_i}, \quad \text{with weight } \alpha_j(p_i), \tag{1.7}$$

where, as described in ([24], p.208), the identification of $\overline{E}_\rho$ with the trivial bundle $\underline{\mathbb{C}}^k$ is given near p_i by

$$\begin{array}{ccc} \bigoplus_{j=1}^{r_i} \mathcal{C}^\infty(\mathcal{U}; \overline{E}^j_{p_i}) & \rightarrow & \mathcal{C}^\infty(\mathcal{U}; \overline{E}_\rho) \\ (\sigma_1, \ldots, \sigma_{r_i}) & \mapsto & \sum_{j=1}^{r_i} \zeta_i^{\alpha_j(p_i)} \sigma_j \end{array} \tag{1.8}$$

with $\zeta_i = e^{2\pi i \sigma_i^{-1} z}$ the complex coordinate introduced earlier. Since ρ is a unitary representation, the bundle $\overline{E}$ comes with a natural Hermitian metric h_E when restricted to the punctured Riemann surface Σ. This Hermitian metric extend to a Hermitian metric $h_{\overline{E}}$ on $\overline{\Sigma}$ which degenerates at the punctures $p_1, \ldots, p_n$. In the trivialization (1.8), it takes the form

$$h_{\overline{E}} \left(\sum_{j=1}^{r_i} \zeta_i^{\alpha_j(p_i)} \sigma_j, \sum_{j=1}^{r_i} \zeta_i^{\alpha_j(p_i)} \sigma_j \right) = \sum_{j=1}^{r_i} |\zeta_i|^{2\alpha_j(p_i)} |\sigma_j|^2 \tag{1.9}$$

The **parabolic degree** of a parabolic bundle is defined by

$$\operatorname{par}\deg(E) = \deg(E) + \sum_{p\in S} \sum_{j=1}^{r(p)} \alpha_j(p) k_j(p),$$

and its **parabolic slope** (again denoted $\mu(E)$) is the ratio of its parabolic degree and its rank. A holomorphic sub-bundle of E with its induced parabolic structure is known as a parabolic sub-bundle of E, and E is said to be a **stable parabolic bundle** if the parabolic slope of any parabolic sub-bundle is strictly smaller than the parabolic slope of E. The Mehta-Seshadri theorem [24] says that a parabolic vector bundle over $\overline{\Sigma}$ arises from an irreducible unitary representation of $\pi_1(\overline{\Sigma} \setminus S)$ if and only if it is parabolically stable and has vanishing parabolic degree.

Biquard [7] proved the analogue of Donaldson's theorem for parabolic bundles, namely, that a parabolic bundle is parabolically stable precisely when it admits a unitary connection with curvature $R = -2\pi i \mu(E)(*1)$, hence recovering the theorem of Mehta-Seshadri when $\mu(E) = 0$. Notice that while Mehta and Seshadri worked with rational weights, Biquard's proof works for arbitrary real weights.

From the Mehta-Seshadri theorem and its generalization by Biquard, the moduli space $\mathcal{N}$ of stable parabolic vector bundles of rank k and parabolic degree zero with prescribed weights and multiplicities at $p_1, \ldots p_n \in \overline{\Sigma}$ is given by

$$\mathcal{N} = \hom(\Gamma, \mathrm{U}(k))^0 / U(k) \tag{1.10}$$

where $\hom(\Gamma, \mathrm{U}(k))^0$ is the space of irreducible admissible representations $\Gamma \to \mathrm{U}(k)$ for the prescribed weights and multiplicities with $\mathrm{U}(k)$ acting on this space by conjugation. Recall that a representation $\rho : \Gamma \to \mathrm{U}(k)$ is **admissible** with respect to a system of weights and multiplicities if the corresponding parabolic vector bundle $\overline{E}_\rho$ has a parabolic structure compatible with this set of weights and multiplicities.

Although $\overline{\Sigma}$ is perhaps the most natural compactification of Σ, our approach will be to consider a different compactification of Σ, also natural, to a manifold with boundary. By replacing each marked point in $\overline{\Sigma}$ with a circle, we keep track of the 'direction' of approach to the cusp. In contrast to $\overline{\Sigma}$, this has the advantage that the natural metric and connection of a stable parabolic vector bundle of degree zero extend non-singularly to the compactification. We will measure regularity in a way adapted to the degeneracy of the hyperboic metric at the cusps by working with a class of adapted differential operators called 'hyperbolic cusp' differential operators.

2. The $\overline{\partial}$-operator for stable parabolic vector bundles

Fix an irreducible representation $\rho : \Gamma \to \mathrm{U}(k)$ with prescribed weights and multiplicities and let $\overline{E} = \overline{E}_\rho$ be the corresponding stable parabolic vector bundle of degree zero. Since $\overline{E}$ is in particular a holomorphic vector bundle, this means there is a $\overline{\partial}$-operator

$$\overline{\partial}_{\overline{E}} : \mathcal{C}^\infty(\overline{\Sigma}; \overline{E}) \to \mathcal{C}^\infty(\overline{E}; \Lambda^{0,1}_{\overline{\Sigma}} \otimes \overline{E}). \tag{2.1}$$

We are interested in the restriction of this operator to the punctured Riemann surface Σ,

$$\overline{\partial}_E : \mathcal{C}^\infty(\Sigma; E) \to \mathcal{C}^\infty(\Sigma; \Lambda^{0,1}_{\Sigma} \otimes E) \tag{2.2}$$

where $E := \overline{E}\big|_\Sigma$ is the restriction of $\overline{E}$ to Σ. We are also interested in the $\overline{\partial}$ operator associated to the endomorphism bundle $\mathcal{E} := \mathrm{End}(E)$,

$$\overline{\partial}_{\mathcal{E}} : \mathcal{C}^\infty(\Sigma; \mathcal{E}) \to \mathcal{C}^\infty(\Sigma; \Lambda^{0,1}_{\Sigma} \otimes \mathcal{E}). \tag{2.3}$$

To study these operators from the point of view of hyperbolic cusp operators (hc-operators), consider the radial blow-up $\widetilde{\Sigma}$ of $\overline{\Sigma}$ at the points $p_1, \ldots, p_n$ with blow-down map

$$\beta : \widetilde{\Sigma} \to \overline{\Sigma}. \tag{2.4}$$

The uniformization theorem for Riemann surfaces specifies a choice of boundary defining function ρ_Σ on $\widetilde{\Sigma}$, that is, a function $\rho_\Sigma \in \mathcal{C}^\infty(\widetilde{\Sigma})$ positive in the interior such that $\rho_\Sigma\big|_{\partial\widetilde{\Sigma}} \equiv 0$ and with $d\rho_\Sigma$ nowhere zero on the boundary. Namely, let g_Σ be the canonical hyperbolic metric in the conformal class specified by the complex structure. Near a point $p \in S$, choose a complex coordinate $\zeta := e^{2\pi i z}$ on $\overline{\Sigma}$ with $\zeta(p) = 0$ such that

$$g_\Sigma = \frac{dx^2 + dy^2}{y^2}, \quad z = x + iy \tag{2.5}$$

in this coordinate. One can use instead the polar coordinates

$$\theta = x, \quad r = \frac{1}{y} \tag{2.6}$$

which also make sense on $\widetilde{\Sigma}$ with $r = 0$ corresponding to the boundary component $\partial\widetilde{\Sigma}_p := \beta^{-1}(p)$. Near $\partial\widetilde{\Sigma}_p$, we choose the boundary defining function ρ_Σ to be given by

$$\rho_\Sigma(\theta, r) = r. \tag{2.7}$$

Doing this near each boundary component and extending ρ_Σ to the interior as a positive function, we get the desired boundary defining function.

Let $\widetilde{\mathbb{H}}$ be the universal cover of $\widetilde{\Sigma}$ so that

$$\widetilde{\Sigma} = \Gamma \setminus \widetilde{\mathbb{H}} \tag{2.8}$$

where we use the canonical identification $\pi_1(\widetilde{\Sigma}) = \pi_1(\Sigma) = \Gamma$. On $\widetilde{\mathbb{H}}$ there is a 'blow-down' map $\beta_{\mathbb{H}} : \widetilde{\mathbb{H}} \to \overline{\mathbb{H}}$ compatible with the action of Γ. Since the action of Γ is free on $\widetilde{\mathbb{H}}$, this means that $\widetilde{\mathbb{H}}$ is in a sense a free resolution of the action of Γ on $\overline{\mathbb{H}}$. In particular, on $\widetilde{\Sigma}$, the pulled back vector bundle $\widetilde{E} := \beta^*\overline{E}$ can be directly described as a quotient of the trivial vector bundle $\widetilde{\mathbb{H}} \times \mathbb{C}^k$ on $\widetilde{\mathbb{H}}$,

$$\widetilde{E} := \Gamma \setminus \left(\widetilde{\mathbb{H}} \times \mathbb{C}^k\right) \tag{2.9}$$

with $\gamma \in \Gamma$ acting on $\widetilde{\mathbb{H}} \times \mathbb{C}^k$ by

$$\begin{array}{rccc} \gamma : & \widetilde{\mathbb{H}} \times \mathbb{C}^k & \to & \widetilde{\mathbb{H}} \times \mathbb{C}^k \\ & (z, v) & \mapsto & (\gamma z, \rho(\gamma) v). \end{array} \tag{2.10}$$

Since ρ is a unitary representation, the canonical Hermitian metric on $\widetilde{\mathbb{H}} \times \mathbb{C}^k$ descends to a Hermitian metric $h_{\widetilde{E}}$ on $\widetilde{E}$. With this metric and the corresponding Chern connection, the vector bundle $\widetilde{E}$ becomes a flat Hermitian vector bundle with holonomy specified by the representation ρ. In particular, it is locally isomorphic to the trivial Hermitian bundle $\underline{\mathbb{C}}^k$. Under such a local identification near the boundary, the operator $\overline{\partial}_{\widetilde{E}}$ can be written as

$$\overline{\partial}_{\widetilde{E}} = \frac{1}{2}\left(\rho_\Sigma d\theta + i\frac{d\rho_\Sigma}{\rho_\Sigma}\right)\left(\frac{1}{\rho_\Sigma}\frac{\partial}{\partial\theta} - i\rho_\Sigma\frac{\partial}{\partial\rho_\Sigma}\right) \tag{2.11}$$

in the polar coordinates (r, θ). The section $\left(\rho_\Sigma d\theta + i\frac{d\rho_\Sigma}{\rho_\Sigma}\right)$ of $\Lambda^{0,1}_\Sigma$ becomes singular as one approaches the boundary. However, it is smooth up to the boundary as a section of ${}^{\mathrm{hc}}\Lambda^{0,1}_{\widetilde{\Sigma}}$, which is defined to be the $(0,1)$ part of the complexified hyperbolic cusp contangent bundle

$$ {}^{\mathrm{hc}}T^*\widetilde{\Sigma} \otimes_{\mathbb{R}} \mathbb{C} = {}^{\mathrm{hc}}\Lambda^{1,0}_{\widetilde{\Sigma}} \oplus {}^{\mathrm{hc}}\Lambda^{0,1}_{\widetilde{\Sigma}}, \tag{2.12}$$

where the bundle ${}^{\mathrm{hc}}T^*\widetilde{\Sigma}$ is defined in such a way that there is a canonical identification

$$\mathcal{C}^\infty(\widetilde{\Sigma};{}^{\mathrm{hc}}T^*\widetilde{\Sigma}) = \{s \in \mathcal{C}^\infty(\widetilde{\Sigma};T^*\widetilde{\Sigma}) \quad | \quad \exists C > 0 \,\text{such that}\; g_\Sigma(s(z),s(z)) \le C \;\forall\, z \in \Sigma\}. \tag{2.13}$$

In that way, $\overline{\partial}_E$ can be seen as a hc-operator (d-operator in the terminology of Vaillant [39]),

$$\overline{\partial}_E : \mathcal{C}^\infty(\widetilde{\Sigma};\widetilde{E}) \to \frac{1}{\rho_\Sigma}\mathcal{C}^\infty(\widetilde{\Sigma};{}^{\mathrm{hc}}\Lambda^{0,1}\widetilde{\Sigma}\otimes\widetilde{E}). \tag{2.14}$$

To keep the discussion short, we refer the reader to the first three sections of [2] for a quick review of hc-operators and a similar construction, and to [39] and [1] for further details.

Similarly, the operator $\overline{\partial}_{\mathcal{E}}$ can be seen as a hc-operator

$$\overline{\partial}_{\mathcal{E}} : \mathcal{C}^\infty(\widetilde{\Sigma};\widetilde{\mathcal{E}}) \to \frac{1}{\rho_\Sigma}\mathcal{C}^\infty(\widetilde{\Sigma};{}^{\mathrm{hc}}\Lambda^{0,1}\widetilde{\Sigma}\otimes\widetilde{\mathcal{E}}). \tag{2.15}$$

where $\widetilde{\mathcal{E}} := \mathrm{End}(\widetilde{E})$. The metric g_Σ induces Hermitian metrics on $\Lambda^{0,1}_\Sigma$ and on ${}^{\mathrm{hc}}\Lambda^{0,1}_{\widetilde{\Sigma}}$. Together with the natural Hermitian metrics $h_{\widetilde{E}}$ and $h_{\widetilde{\mathcal{E}}}$, this therefore defines Hilbert spaces $\mathcal{H}_{\widetilde{E},j} := \mathrm{L}^2(({}^{\mathrm{hc}}\Lambda^{0,1})^j\otimes\widetilde{E})$ and $\mathcal{H}_{\widetilde{\mathcal{E}},j} := \mathrm{L}^2(({}^{\mathrm{hc}}\Lambda^{0,1})^j\otimes\widetilde{\mathcal{E}})$ for $j\in\{0,1\}$ with inner product given by

$$\begin{aligned}\langle f_1,f_2\rangle_{\mathcal{H}_{\widetilde{E},j}} &:= \int_{\widetilde{\Sigma}}\langle f_1(z),f_2(z)\rangle_{({}^{\mathrm{hc}}\Lambda^{0,1})^j\otimes\widetilde{E}}\,dg_\Sigma(z);\\ \langle f_1,f_2\rangle_{\mathcal{H}_{\widetilde{\mathcal{E}},j}} &:= \int_{\widetilde{\Sigma}}\langle f_1(z),f_2(z)\rangle_{({}^{\mathrm{hc}}\Lambda^{0,1})^j\otimes\widetilde{\mathcal{E}}}\,dg_\Sigma(z);\end{aligned} \tag{2.16}$$

where dg_Σ is the natural extension of the volume form of g_Σ to $\widetilde{\Sigma}$. With these inner products, we can define the formal adjoints $\overline{\partial}^*_E$ and $\overline{\partial}^*_{\mathcal{E}}$ of $\overline{\partial}_E$ and $\overline{\partial}_{\mathcal{E}}$. Recall then that the operators

$$D_E := \sqrt{2}(\overline{\partial}_E + \overline{\partial}^*_E), \quad D_{\mathcal{E}} := \sqrt{2}(\overline{\partial}_{\mathcal{E}} + \overline{\partial}^*_{\mathcal{E}}) \tag{2.17}$$

can be interpreted as Dirac type hc-operators acting on the Clifford modules $(\underline{\mathbb{C}}\oplus{}^{\mathrm{hc}}\Lambda^{0,1}_\Sigma)\otimes\widetilde{E}$ and $(\underline{\mathbb{C}}\oplus{}^{\mathrm{hc}}\Lambda^{0,1}_\Sigma)\otimes\widetilde{\mathcal{E}}$ with Clifford action given by

$$c(f) = \sqrt{2}(\varepsilon(f^{0,1}) - \iota(f^{1,0})), \quad f\in\mathcal{C}^\infty(\widetilde{\Sigma};{}^{\mathrm{hc}}\Lambda_{\widetilde{\Sigma}}) \tag{2.18}$$

where $\varepsilon(f^{0,1})$ denotes exterior multiplication by the form $f^{0,1}$. In his thesis, Vaillant provided a criterion to determine when a Dirac type hc-operator $\eth_{\mathrm{hc}}$ is Fredholm. For a Dirac type hc-operator $\eth_{\mathrm{hc}}$ acting on $\mathcal{C}^\infty(\widetilde{\Sigma};W)$, the criterion is relatively easy to formulate. One first needs to introduce the vertical operator

$$\eth^V_{\mathrm{hc}} := \rho_\Sigma\eth_{\mathrm{hc}}|_{\partial\widetilde{\Sigma}} \tag{2.19}$$

on the boundary $\widetilde{\Sigma}$. When $\ker \eth^V_{\mathrm{hc}}$ is non-trivial, one can also introduce a horizontal operator

$$\eth^H_{\mathrm{hc}} : \ker \eth^V_{\mathrm{hc}} \to \ker \eth^V_{\mathrm{hc}} \tag{2.20}$$

defined by

$$\eth^H_{\mathrm{hc}} \xi = \Pi_0 (\eth_{\mathrm{hc}} \widetilde{\xi} \big|_{\partial \widetilde{\Sigma}}) \tag{2.21}$$

where $\xi \in \ker \eth^V_{\mathrm{hc}}$, $\widetilde{\xi} \in \mathcal{C}^\infty(\widetilde{\Sigma}; W)$ is a smooth extension of ξ in the interior and Π_0 is the orthogonal projection from $\mathrm{L}^2(\partial\widetilde{\Sigma}; W)$ to $\ker \eth^V_{\mathrm{hc}}$. In terms of these operators, the criterion of Vaillant can be formulated as follows.

Proposition 2.1 (Vaillant [39], §3)**.** *The continuous spectrum of a Dirac type self-adjoint operator $\eth_{\mathrm{hc}} \in \Psi^1_{\mathrm{hc}}(\widetilde{\Sigma}; W)$ is governed by the spectrum of the horizontal family $\eth^H_{\mathrm{hc}}$ with bands of continuous spectrum starting at the eigenvalues of $\eth^H_{\mathrm{hc}}$ and going to infinity. In particular, $\eth_{\mathrm{hc}}$ is Fredhom if and only if $\eth^H_{\mathrm{hc}}$ is invertible and $\eth_{\mathrm{hc}}$ is Fredholm and has discrete spectrum whenever $\eth^V_{\mathrm{hc}}$ is invertible.*

To apply this criterion to D_E and $D_{\mathcal{E}}$, we first need to describe the restriction of $\widetilde{E}$ on the boundary $\partial\widetilde{\Sigma}$. Let us fix a boundary component $\partial\widetilde{\Sigma}_{p_i}$ for some $p_i \in S$. Identify $\partial\widetilde{\Sigma}_{p_i}$ with $\mathbb{Z}\setminus\mathbb{R}$ as oriented manifolds. Notice that the orientation of $\partial\widetilde{\Sigma}_{p_i}$ induced from $\widetilde{\Sigma}$ is such that $\frac{\partial}{\partial u}$ is an oriented section of the tangent bundle where $u = -x = -\theta$ in terms of the polar coordinates (r, θ). From that perspective, we can interpret the restriction $\widetilde{E}_i := \widetilde{E}\Big|_{\partial\widetilde{\Sigma}_{p_i}}$ as the quotient

$$\mathbb{Z} \setminus (\mathbb{R} \times \mathbb{C}^k) \tag{2.22}$$

where $\mathbb{R} \times \mathbb{C}^k$ is the total space of the trivial vector bunlde $\underline{\mathbb{C}}^k$ over $\mathbb{R}$ on which $\mathbb{Z}$ acts by

$$\begin{array}{rccc} m: & \mathbb{R} \times \mathbb{C}^k & \to & \mathbb{R} \times \mathbb{C}^k \\ & (u, v) & \mapsto & (u + m, \rho(S_i)^{-m} v) \end{array} \tag{2.23}$$

for $m \in \mathbb{Z}$. Sections of $\widetilde{E}_i$ then correspond to $\mathbb{Z}$-invariant sections of the trivial bundle $\underline{\mathbb{C}}^k \to \mathbb{R}$. On the other hand, at the boundary component $\partial\widetilde{\Sigma}_{p_i}$, the vertical family of D_E is given by

$$D^V_{\widetilde{E}_i} := \rho_\Sigma D_E|_{\partial\widetilde{\Sigma}_{p_i}} = c(du)\frac{\partial}{\partial u} \tag{2.24}$$

acting on $(\underline{\mathbb{C}} \oplus {}^{\mathrm{hc}}\Lambda^{0,1}_{\widetilde{\Sigma}}) \otimes \widetilde{E}_i$. Under the standard identification

$$-c\left(\frac{d\rho_\Sigma}{\rho_\Sigma}\right) : {}^{\mathrm{hc}}\Lambda^{0,1}_{\widetilde{\Sigma}}\Big|_{\partial\widetilde{\Sigma}_{p_i}} \otimes \widetilde{E}_i \to \widetilde{E}_i \tag{2.25}$$

given by Clifford multiplication by the element $-c\left(\frac{d\rho_\Sigma}{\rho_\Sigma}\right)$, we can rewrite the operator as

$$D^V_{\widetilde{E}_i} = \begin{pmatrix} 0 & \overline{\partial}^V_{\widetilde{E}_i} \\ \overline{\partial}^V_{\widetilde{E}_i} & 0 \end{pmatrix} = \begin{pmatrix} 0 & i\frac{\partial}{\partial u} \\ i\frac{\partial}{\partial u} & 0 \end{pmatrix} \in \Psi^1(\partial\widetilde{\Sigma}_{p_i}; \widetilde{E}_i \oplus \widetilde{E}_i). \tag{2.26}$$

Lemma 2.2. *The spectrum of the vertical family $\overline{\partial}^V_{\widetilde{E}_i}$ is given by*

$$\lambda_{j,k} = 2\pi(\alpha_j(p_i) + k), \quad j \in \{1, \dots, r_i\}, \quad k \in \mathbb{Z}$$

where the eigenvalue $\lambda_{j,k}$ has multiplicity $k_j(p_i)$.

Proof. Let

$$\mathbb{C}^k = \bigoplus_{j=1}^{r_i} W_{ij} \tag{2.27}$$

be the decomposition in terms of the eigenspaces of $\rho(S_i)$ where W_{ij} is the eigenspace corresponding to the eigenvalue $e^{2\pi i\alpha_j(p_i)}$. As an operator on $\mathbb{R}$, $\overline{\partial}^V_{\widetilde{E}_i}$ commutes with the action of $\mathbb{Z}$. This means that $\overline{\partial}^V_{\widetilde{E}_i}$ decomposes as a sum of operators

$$\overline{\partial}^V_{\widetilde{E}_i} = \bigoplus_{j=1}^{r_i} \overline{\partial}^V_{\widetilde{E}_{ij}} \tag{2.28}$$

with $\overline{\partial}^V_{\widetilde{E}_{ij}}$ acting on sections of

$$\widetilde{E}_{ij} := \mathbb{Z} \setminus (\mathbb{R} \times W_{ij}). \tag{2.29}$$

If $w^i_1, \dots, w^i_{k_j}$ is a basis of W_{ij}, then clearly

$$e^{ij}_{kl} = e^{-2\pi i(k+\alpha_j(p_i))u} w^i_l, \quad k \in \mathbb{Z}, \quad l \in \{1, \dots k_j(p_i)\}, \tag{2.30}$$

is a basis of $\mathrm{L}^2(\partial\widetilde{\Sigma}_{p_i}; \widetilde{E}_{ij})$ in terms of eigensections of $\overline{\partial}^V_{\widetilde{E}_{ij}}$ with e^{ij}_{kl} an eigensection with eigenvalue $\lambda_{j,k}$. Collecting these eigenvalues for all j, the result follows. □

Proposition 2.3. *The operator D_E is Fredholm. Moreover, if none of the weights of the parabolic structure of $\overline{E}$ are zero, then its spectrum is discrete.*

Proof. This will follows from proposition 2.1. First, if none of the weights are zero, then it follows from lemma 2.2 that the vertical family D^V_E is invertible, so that D_E is Fredholm with discrete spectrum. If one of the weights is zero at some point, say $\alpha_1(p_i) = 0$ for some $p_i \in S$, we need to check that the horizontal family

$$D^H_{\widetilde{E}_i} : \ker D^V_{\widetilde{E}_i} \to \ker D^V_{\widetilde{E}_i} \tag{2.31}$$

is invertible to insure that D_E is Fredholm. This will follow from the following lemma. □

Lemma 2.4. *If $\alpha_1(p_i)=0$, then the horizontal family $D^H_{\widetilde{E}_i}$ is given by*

$$D^H_{\widetilde{E}_i} = -\frac{1}{2} ic(du) : \ker D^V_{\widetilde{E}_i} \to \ker D^V_{\widetilde{E}_i}. \tag{2.32}$$

In particular, it is invertible.

Proof. Since $\widetilde{E}$ is flat, the proof is essentially the same as in ([2], Proposition 3.1) with $\ell = 0$. Notice first that the bundle on which D_E acts is $(\underline{\mathbb{C}}^k \otimes {}^{\mathrm{hc}}\Lambda^{0,1}_{\widetilde{\Sigma}}) \otimes \widetilde{E}$. Choose a spin structure on $\widetilde{\Sigma}$ and let S be the corresponding spinor bundle with respect to the metric g_Σ. Seen as a complex line bundle, S is a square root of the canonical line bundle,

$$S \otimes_{\mathbb{C}} S = K \tag{2.33}$$

Moreover, we have that

$$(\underline{\mathbb{C}} \oplus {}^{\mathrm{hc}}\Lambda^{0,1}_{\widetilde{\Sigma}}) \cong S \otimes_{\mathbb{R}} S^* \tag{2.34}$$

as real vector bundles. Thus, the operator D_E acts on

$$S \otimes_{\mathbb{R}} (S^* \otimes_{\mathbb{C}} \widetilde{E}).$$

As a bundle with connection, the bundle $S^* \otimes_{\mathbb{C}} \widetilde{E}$ certainly does not have a product structure near the boundary since it has non-zero curvature. According to proposition 3.15 in [39], the horizontal operator $D^H_{\widetilde{E}_i}$ is given by

$$(-iR)c\left(\frac{\partial}{\partial u}\right) \tag{2.35}$$

where $iRdg_\Sigma$ is the curvature of the complex vector bundle $S^* \otimes \widetilde{E}$, really the curvature of S^* since $\widetilde{E}$ is flat. Since S^* is a square root of K^{-1}, this means $R = \frac{1}{2}$ and the result follows. □

For the operator $\overline{\partial}_{\mathcal{E}}$, there is a similar discussion. The restriction $\widetilde{\mathcal{E}}_i$ of the endomorphism bundle $\widetilde{\mathcal{E}}$ to $\partial\widetilde{\Sigma}_{p_i}$ can be described as the quotient

$$\mathbb{Z} \setminus (\mathbb{R} \times M_{k\times k}(\mathbb{C})) \tag{2.36}$$

where $\mathbb{R} \times M_{k\times k}(\mathbb{C})$ is the total space of the trivial bundle of $k \times k$ complex matrices over $\mathbb{R}$ with $\mathbb{Z}$ action given by

$$\begin{array}{rccc} m: & \mathbb{R} \times M_{k\times k}(\mathbb{C}) & \to & \mathbb{R} \times M_{k\times k}(\mathbb{C}) \\ & (u, A) & \mapsto & (u+m, \rho(S_i)^{-m} A \rho(S_i)^m) \end{array} \tag{2.37}$$

for $m \in \mathbb{Z}$. The identification

$$-c\left(\frac{d\rho_\Sigma}{\rho_\Sigma}\right) : {}^{\mathrm{hc}}\Lambda^{0,1}_{\Sigma}\Big|_{\partial\widetilde{\Sigma}_{p_i}} \otimes \widetilde{\mathcal{E}}_i \to \widetilde{\mathcal{E}}_i \tag{2.38}$$

given by Clifford multiplication then allows one to write the vertical operator $D^V_{\widetilde{\mathcal{E}}_i}$ as

$$D^V_{\widetilde{\mathcal{E}}_i} = \begin{pmatrix} 0 & \overline{\partial}^V_{\widetilde{\mathcal{E}}_i} \\ \overline{\partial}^V_{\widetilde{\mathcal{E}}_i} & 0 \end{pmatrix} = \begin{pmatrix} 0 & i\frac{\partial}{\partial u} \\ i\frac{\partial}{\partial u} & 0 \end{pmatrix} \in \Psi^1(\partial\widetilde{\Sigma}_{p_i}; \widetilde{\mathcal{E}}_i \oplus \widetilde{\mathcal{E}}_i). \tag{2.39}$$

Lemma 2.5. *The spectrum of the vertical family* $\overline{\partial}^V_{\widetilde{\mathcal{E}}_i}$ *is given by*

$$\lambda_k(j,l) = 2\pi(k + \alpha_j(p_i) - \alpha_l(p_i)) \quad \textit{with multiplicity } k_j(p_i)k_l(p_i)$$

for $k \in \mathbb{Z}$ *and* $j, l \in \{1, \cdots, r_i\}$.

Proof. In terms of the decomposition (2.27), we have the decomposition

$$M_{k\times k}(\mathbb{C}) = \mathbb{C}^k \otimes (\mathbb{C}^k)^* = \bigoplus_{j,l=1}^{r_i} W_{ij} \otimes W_{il}^* \tag{2.40}$$

into the eigenspaces of the adjoint action of $\rho(S_i)$ on $M_{k\times k}(\mathbb{C})$. Here, the eigenspace $W_{ij} \otimes W_{il}^*$ has corresponding eigenvalue $e^{2\pi i(\alpha_j - \alpha_l)}$. With respect to these eigenspaces, the vertical operator decomposes as

$$\overline{\partial}^V_{\mathcal{E}_i} = \bigoplus_{j,l=1}^{r_i} \overline{\partial}^V_{\hom(\widetilde{E}_{il};\widetilde{E}_{ij})} \tag{2.41}$$

with $\overline{\partial}^V_{\hom(\widetilde{E}_{il};\widetilde{E}_{ij})}$ acting on sections of

$$\hom(\widetilde{E}_{il}, \widetilde{E}_{ij}) := \mathbb{Z} \setminus (\mathbb{R} \times (W_{ij} \otimes W_{il}^*)). \tag{2.42}$$

If $f_1, \ldots, f_m$ form a basis of $W_{ij} \otimes W_{il}^*$, then

$$e_{kp}^{ijl} := e^{-2\pi i(k+\alpha_j-\alpha_l)u} f_p, \quad p \in \{1, \ldots, m\},\ k \in \mathbb{Z} \tag{2.43}$$

will be a $\overline{\partial}^V_{\hom(\widetilde{E}_{il},\widetilde{E}_{ij})}$-eigenbasis of $\mathrm{L}^2(\partial\widetilde{\Sigma}_{p_i}, \hom(\widetilde{E}_{il};\widetilde{E}_{ij}))$ with e_{kp}^{ijl} having eigenvalue $\lambda_k(j,l)$, from which the result follows. $\square$

Proposition 2.6. *The operator* $\overline{\partial}_{\mathcal{E}}$ *is Fredholm.*

Proof. As in lemma 2.4, one computes that the horizontal operator of $D_{\mathcal{E}}$ at the point p_i is given by

$$D^H_{\widetilde{\mathcal{E}}_i} = -\frac{i}{2}c(du) : \ker D^V_{\widetilde{\mathcal{E}}_i} \to \ker D^V_{\widetilde{\mathcal{E}}_i}. \tag{2.44}$$

In particular, it is clearly invertible and the result follows from proposition 2.1. $\square$

To end this section, let us compute the eta invariants of the self-adjoint operators $\overline{\partial}^V_{\widetilde{E}_{ij}}$ and $\overline{\partial}^V_{\hom(\widetilde{E}_{il},\widetilde{E}_{ij})}$.

Lemma 2.7. *The eta invariants of* $\overline{\partial}^V_{\widetilde{E}_{ij}}$ *and* $\overline{\partial}^V_{\hom(\widetilde{E}_{il},\widetilde{E}_{ij})}$ *defined in* (2.28) *and* (2.41) *are given by*

$$\eta(\overline{\partial}^V_{\widetilde{E}_{ij}}) = \begin{cases} k_j^i(1 - 2\alpha_j^i), & \alpha_j^i > 0, \\ 0, & \alpha_j^i = 0, \end{cases}$$

$$\eta(\overline{\partial}^V_{\hom(\widetilde{E}_{il},\widetilde{E}_{ij})}) = \begin{cases} k_j^i k_l^i \operatorname{sign}(\alpha_j^i - \alpha_l^i)(1 - 2|\alpha_j^i - \alpha_l^i|), & j \neq l, \\ 0, & j = l, \end{cases}$$

where $\alpha_j^i := \alpha_j(p_i)$ *and* $k_j^i = k_j(p_i)$.

Proof. The eta invariant of $\overline{\partial}^V_{\widetilde{E}_{ij}}$ is the value at $s=0$ of the meromorphic extension of the eta functional

$$\eta(\overline{\partial}^V_{E_{ij}}, s) := \sum_{\lambda\in \operatorname{spec}(\overline{\partial}^V_{E_{ij}})\setminus\{0\}} \frac{\lambda}{|\lambda|^{s+1}}, \quad \operatorname{Re} s >> 1. \tag{2.45}$$

According to (2.30) the spectrum of $\overline{\partial}^V_{E_{ij}}$ is symmetric when $\alpha_j(p_i)=0$, so in that case the eta invariant vanishes. When $\alpha_j>0$, we get instead for $\operatorname{Re} s \gg 0$

$$\begin{aligned}\eta(\overline{\partial}^V_{E_{ij}}, s) &= \frac{k_j(p_i)}{(2\pi)^s}\sum_{k\in\mathbb{Z}} \frac{k+\alpha_j(p_i)}{|k+\alpha_j(p_i)|^{s+1}} \\ &= \frac{k_j(p_i)}{(2\pi)^s}\left(\sum_{k=0}^{\infty}\frac{1}{|k+\alpha^i_j|^s} - \sum_{k=1}^{\infty}\frac{1}{|k-\alpha^i_j|^s}\right) \\ &= \frac{k_j(p_i)}{(2\pi)^s}(\zeta_H(s,\alpha_j(p_i)) - \zeta_H(s, 1-\alpha_j(p_i)))\end{aligned} \tag{2.46}$$

where

$$\zeta_H(s,\beta) = \sum_{k=0}^{\infty}\frac{1}{|k+\beta|^s}, \quad \operatorname{Re} s > 1, \tag{2.47}$$

is the Hurwitz zeta function. It admits an analytic continuation to $\mathbb{C}\setminus\{1\}$ and its value at $s=0$ is given by

$$\zeta(0,\beta) = \frac{1}{2} - \beta, \quad \text{when } \beta > 0. \tag{2.48}$$

Thus, this gives

$$\begin{aligned}\eta(\overline{\partial}^V_{E_{ij}}) &= k_j(p_i)(\zeta_H(0,\alpha_j(p_i)) - \zeta_H(0,1-\alpha_j(p_i))) \\ &= k_j(p_i)(1-2\alpha_j(p_i))\end{aligned} \tag{2.49}$$

as claimed. For $\overline{\partial}^V_{\hom(E_{il},E_{ij})}$, the computation is very similar and relies on the knowledge of its spectrum described in (2.43). We leave the details to the reader. □

3. Families of $\overline{\partial}$-operators over the moduli space

For the moduli space $\mathcal{N}$, a **universal parabolic stable vector bundle** is a Hermitian vector bundle $E \to \Sigma\times\mathcal{N}$ such that for each $[\rho]\in\mathcal{N}$,

$$E|_{\Sigma\times\{[\rho]\}} \cong E_\rho \tag{3.1}$$

as Hermitian vector bundles. From the point of view of representation theory, the existence of a universal parabolic stable vector bundle is equivalent to the existence of a smooth section for the smooth principal $\mathrm{PU}(k)$-bundle

$$\hom(\Gamma, \mathrm{U}(k))^0 \to \hom(\Gamma,\mathrm{U}(k))^0/\,\mathrm{U}(k). \tag{3.2}$$

According to ([12], Proposition 3.2), a universal parabolic stable vector bundle exists for a generic weight system. From deformation theory, we know also that for $[\rho] \in \mathcal{N}$, there exists a small neighborhood $\mathcal{U} \subset \mathcal{N}$ of $[\rho]$ such that there exists a universal parabolic stable vector bundle on $\Sigma \times \mathcal{U}$. From now on, we will assume either that the moduli space $\mathcal{N}$ admits a universal parabolic vector bundle $E \to \Sigma \times \mathcal{N}$ or else, that we restrict $\mathcal{N}$ to an open set $\mathcal{U}$ admitting a universal parabolic stable vector bundle. Let

$$\sigma : \mathcal{N} \to \hom(\Gamma, U(k))^0 \tag{3.3}$$

be a smooth section and consider the induced universal parabolic vector bundle $E \to \Sigma \times \mathcal{N}$ such that

$$E|_{\Sigma\times\{[\rho]\}} = E_{\sigma([\rho])}, \quad [\rho] \in \mathcal{N}. \tag{3.4}$$

On each fibre $\pi^{-1}([\rho])$ of the holomorphic fibration $\pi : \Sigma \times \mathcal{N} \to \mathcal{N}$, consider the operators $\overline{\partial}_{E_{\sigma([\rho])}}$ and $\overline{\partial}_{\mathcal{E}_{\sigma([\rho])}}$. They combine to give families of $\overline{\partial}$ hc-operators

$$\overline{\partial}_E \in \Psi^1_{\mathrm{hc}}(\Sigma \times \mathcal{N}/\mathcal{N}; E, {}^{\mathrm{hc}}\Lambda^{0,1}_\Sigma \otimes E), \quad \overline{\partial}_{\mathcal{E}} \in \Psi^1_{\mathrm{hc}}(\Sigma \times \mathcal{N}/\mathcal{N}; \mathcal{E}, {}^{\mathrm{hc}}\Lambda^{0,1}_\Sigma \otimes \mathcal{E}) \tag{3.5}$$

parametrized by $\mathcal{N}$. Since the fibration π is trivial, it has a natural choice of connection, namely the trivial one. Since E is a Hermitian vector bundle, it has also a natural connection, namely its Chern connection. To describe it, we will first discuss the equivalent of the Bers coordinates for the moduli space $\mathcal{N}$ as introduced in [38] (see also [40] for similar coordinates on the moduli space of stable vector bundles on a compact Riemann surface).

Recall first that the holomorphic tangent space $T_{[\rho]}\mathcal{N}$ is naturally identified with the space $\mathcal{H}^{0,1}(\Sigma, \mathrm{End}(E_\rho))$ of square integrable harmonic $(0,1)$-forms on Σ with value in $\mathrm{End}(E_\rho)$. From the point of view of index theory, this is intuitively clear. An infinitesimal deformation of the holomorphic structure of E_ρ corresponds to an infinitesimal deformation of the operator $\overline{\partial}_{E_\rho}$, which amounts to adding an infinitesimal $(0,1)$-form $\nu \in \dot{\mathcal{C}}^\infty(\Sigma; \Lambda^{0,1}\Sigma \otimes \mathrm{End}(E_\rho))$,

$$\overline{\partial}_{E_\rho} \to \overline{\partial}_{E_\rho} + \nu. \tag{3.6}$$

However, if ν is in the image of $\overline{\partial}_{\mathcal{E}_\rho}$, say $\nu = \overline{\partial}_{\mathcal{E}_\rho}\mu$, then ν is obtained from the infinitesimal reparametrization of E given by

$$\mathrm{Id}_{E_\rho} + \mu : E_\rho \to E_\rho, \quad \overline{\partial}_{E_\rho} + \nu = (\mathrm{Id}_{E_\rho} - \mu)\overline{\partial}_{E_\rho}(\mathrm{Id}_E + \mu) = \overline{\partial}_E + \overline{\partial}_{\mathcal{E}_\rho}\mu. \tag{3.7}$$

In this case, the deformation ν leads to the same holomorphic structure up to biholomorphism. To get deformations leading to new holomorphic structures up to biholomorphism, we need to mod out by the image of $\overline{\partial}_{\mathcal{E}_\rho}$. This means we can identify $T_{[\rho]}\mathcal{N}$ with

$$\mathrm{coker}\, \overline{\partial}_{\mathcal{E}_\rho} \cong \mathcal{H}^{0,1}(\Sigma; \mathrm{End}(E_\rho)). \tag{3.8}$$

The natural non-degenerate pairing

$$\begin{array}{ccc} \mathcal{H}^{0,1}(\Sigma;\operatorname{End}(E_\rho))\otimes\mathcal{H}^{1,0}(\Sigma;\operatorname{End}(E_\rho)) & \to & \mathbb{C} \\ (\nu,\theta) & \mapsto & \int_\Sigma \operatorname{tr}(\nu\wedge\theta) \end{array} \tag{3.9}$$

allows one to identify $T^*_{[\rho]}\mathcal{N}$ with the space $\mathcal{H}^{1,0}(\Sigma;\operatorname{End}(E_\rho))$ of square integrable harmonic $(1,0)$-forms on Σ with values in $\operatorname{End}(E_\rho)$.

Let $\rho:\Gamma\to U(k)$ be an admissible representation and suppose that $\rho=\sigma([\rho])$ where σ is the smooth section (3.3) defining the universal stable parabolic vector bundle $E\to\Sigma\times\mathcal{N}$. As pointed out in [38], for each $\nu\in\mathcal{H}^{0,1}(\Sigma;\operatorname{End}(E_\rho))$ small enough, there exists a unique map $f^\nu:\mathbb{H}\to\operatorname{GL}(k,\mathbb{C})$ such that

(i) $\frac{\partial f^\nu}{\partial\bar z}(z)=f^\nu(z)\nu(z),\quad z\in\mathbb{H}$, where we also write ν for the lift of ν to the universal cover $\mathbb{H}$ of Σ;
(ii) $\rho^\nu(\gamma):=f^\nu(\gamma z)\rho(\gamma)f^\nu(z)^{-1}$ is independent of $z\in\mathbb{H}$ and is an admissible irreducible unitary representation of Γ with $\rho^\nu=\sigma([\rho^\nu])$.
(iii) f^ν is regular at the cusps, that is,

$$f^\nu(x_i)=\lim_{z\to\infty}f^\nu(\sigma_i z)<\infty, i=1,\dots,n.$$

(iv) $\det(f^\nu(z_0))=1$ at $z_0=\sqrt{-1}\in\mathbb{H}$.
(v) For $\epsilon\in[0,1)$, there are solutions $f^{\epsilon\nu}$ to (i),(ii),(iii) and (iv) with ν replaced with $\epsilon\nu$ in such a way that $f^0=\operatorname{Id}$ is the identity section and

$$[0,1]\ni\epsilon\mapsto f^{\epsilon\nu}(z_0)\in\operatorname{GL}(k,\mathbb{C})$$

is a continuous map.

For a fixed ν, it not hard to see that there is at most one isomorphism class $[\rho^\nu]\in\mathcal{N}$ of irreducible admissible representations such that these properties are satisfied by some f^ν. Since we require that $\rho^\nu=\sigma([\rho^\nu])$, one can also check that requirements (i)-(iii) determine the solution f^ν up to multiplication by an element of $U(1)\subset U(k)$. Requirement (iv) further forces the solution to be unique up to multiplication by an element of $\mathbb{Z}_k\subset U(1)\subset U(k)$. Since $\mathbb{Z}_k$ is a discrete subgroup of $U(k)$, the last requirement chooses canonically a unique solution among these k possible solutions.

These solutions allow us to introduce complex coordinates near $[\rho]$ in $\mathcal{N}$. Precisely, if $\nu_1,\dots,\nu_d$ form a basis of $\mathcal{H}^{0,1}(\Sigma,E_{\sigma([\rho])})$, then we get complex coordinates

$$\mathbb{C}^d\ni(\varepsilon_1,\dots,\varepsilon_d)\mapsto[\rho^\nu]\in\mathcal{N}, \tag{3.10}$$

where $\nu=\varepsilon_1\nu_1+\cdots+\varepsilon_d\nu_d$. As mentioned in [38], the complex coordinates introduced in this way at two different points $[\rho_1],[\rho_2]\in\mathcal{N}$ transform holomorphically on overlaps, which induces on $\mathcal{N}$ a complex structure.

Now, for each $[\rho]\in\mathcal{N}$, we can introduce these coordinates. This allows us to define a canonical connection on the universal stable parabolic vector bundle $E\to\Sigma\times\mathcal{N}$. On each fibre

$$E|_{\Sigma\times\{[\rho]\}}=E_{\sigma([\rho])}, \tag{3.11}$$

the canonical connection restricts to the flat connection on $E_{\sigma([\rho])}$. In the horizontal direction at $(z,[\rho]) \in \Sigma \times \mathcal{N}$, we use the complex coordinates $\nu \in \mathcal{H}^{0,1}(\Sigma, \operatorname{End}(E_\rho))$ introduced above and define

$$\begin{aligned} \nabla_\nu e(z,[\rho]) &:= \frac{\partial}{\partial \varepsilon}(f^{\varepsilon\nu})^{-1} e(z,[\rho^{\varepsilon\nu}])\Big|_{\varepsilon=0}, \\ \nabla_{\overline{\nu}} e(z,[\rho]) &:= \frac{\partial}{\partial \overline{\varepsilon}}(f^{\varepsilon\nu})^{-1} e(z,[\rho^{\varepsilon\nu}])\Big|_{\varepsilon=0}. \end{aligned} \tag{3.12}$$

These combine to give a canonical connection ∇^E for the universal stable parabolic vector bundle E. One can also check that together with the family $\overline{\partial}_E$, (3.12) induces a holomorphic structure on E. This connection and holomorphic structure also naturally induce a connection $\nabla^{\mathcal{E}}$ and a holomorphic structure on the endomorphism bundle $\mathcal{E} = \operatorname{End}(E)$.

Lemma 3.1. *The connections ∇^E and $\nabla^{\mathcal{E}}$ are the respective Chern connections of the Hermitian bundles $E \to \Sigma \times \mathcal{N}$ and $\mathcal{E} \to \Sigma \times \mathcal{N}$.*

Proof. We will proceed as in the proof of lemma 1 in [40]. Let $e_1(z,[\rho^\nu])$ and $e_2(z,[\rho^\nu])$ be local sections of E near $(z_0,[\rho]) \in \Sigma \times \mathcal{N}$ and set

$$\widetilde{e}_i(z,[\rho^\nu]) := (f^\nu)^{-1} e_i(z,[\rho^\nu]). \tag{3.13}$$

From

$$\langle e_1(z,[\rho^\nu]), e_2(z,[\rho^\nu])\rangle_{E_{(z,[\rho^\nu])}} = \langle \overline{f^\nu}^\dagger f^\nu \widetilde{e}_1(z,[\rho^\nu]), \widetilde{e}_2(z,[\rho^\nu])\rangle_{E_{(z,[\rho])}}, \tag{3.14}$$

we see that ∇^E will be the Chern connection of E provided

$$\Phi_\nu := \frac{\partial}{\partial \varepsilon} \overline{f^{\varepsilon\nu}}^\dagger f^{\varepsilon\nu}\Big|_{\varepsilon=0} = 0 \tag{3.15}$$

for each $[\rho] \in \mathcal{N}$ and $\nu \in \mathcal{H}^{0,1}(\Sigma; \operatorname{End}(E_{\sigma([\rho])})$. From property (ii) of the definition of $f^{\varepsilon\nu}$, we see that $\Phi_\nu \in \Omega^0(\Sigma, \operatorname{End}(E_{\sigma([\rho])}))$. In fact, a simple computation shows that $\partial_{\overline{z}}\Phi_\nu = 0$, so that $\Phi_\nu \in \ker \overline{\partial}_{\mathcal{E}_{\sigma([\rho])}}$. Since $E_{\sigma([\rho]}$ is stable, this means that Φ_ν is a multiple of the identity, and from the normalization condition (iv) of the definition of $f^{\varepsilon\nu}$, we conclude that $\Phi_\nu = 0$.

Similarly, the connection $\nabla^{\mathcal{E}}$ will be the Chern connection of $\mathcal{E}$ provided $\operatorname{Ad}\Phi_\nu = 0$ for all $[\rho] \in \mathcal{N}$ and $\nu \in \mathcal{H}^{0,1}(\Sigma, \operatorname{End}(E_{\sigma([\rho])}))$, which follows immediatly from (3.15). □

Since the fibration $\Sigma \times \mathcal{N} \to \mathcal{N}$ is trivial, we can choose to put on it the trivial connection. with this choice and the connection ∇^E, we can then take the covariant derivatives for a family of operators $A \in \Psi^*(\Sigma \times \mathcal{N}/\mathcal{N}; E)$ with respect to horizontal directions on $\mathcal{N}$. For the family of operators $\overline{\partial}_E$, one can compute explicitly that

$$\begin{aligned} \nabla_\nu \overline{\partial}_E &= \nu, & \nabla_{\overline{\nu}} \overline{\partial}_E &= 0, \\ \nabla_\nu \overline{\partial}_E^* &= 0, & \nabla_{\overline{\nu}} \overline{\partial}_E^* &= - * \nu * \end{aligned} \tag{3.16}$$

at $[\rho]\in\mathcal{N}$, where $\nu\in\mathcal{H}^{0,1}(\Sigma,\mathrm{End}(E_{\sigma([\rho])}))$. Similarly, for the family of operators $\overline{\partial}_{\mathcal{E}}$, one computes (cf. p.127 in [38])

$$\begin{aligned} \nabla_\nu\overline{\partial}_{\mathcal{E}} &= \mathrm{ad}\,\nu, & \nabla_{\overline{\nu}}\overline{\partial}_{\mathcal{E}} &= 0,\\ \nabla_\nu\overline{\partial}^*_{\mathcal{E}} &= 0, & \nabla_{\overline{\nu}}\overline{\partial}^*_{\mathcal{E}} &= -*\mathrm{ad}(*\nu) \end{aligned} \tag{3.17}$$

at $[\rho]\in\mathcal{N}$, where $\nu\in\mathcal{H}^{0,1}(\Sigma,\mathrm{End}(E_{\sigma([\rho])}))$.

The canonical connection ∇^E also naturally extends to define a connection on $\widetilde{E}\to\widetilde{\Sigma}\times\mathcal{N}$. When we consider $\overline{\partial}_E$ and $\overline{\partial}_{\mathcal{E}}$ as families of hc-operators, this leads to the following useful fact.

Lemma 3.2. *The vertical families $D^V_{\widetilde{E}_i}$ and $D^V_{\widetilde{\mathcal{E}}_i}$ are parallel with respect to the induced connection,*

$$[\nabla^{\widetilde{E}_i\oplus\widetilde{E}_i}, D^V_{\widetilde{E}_i}]=0,\quad [\nabla^{\widetilde{\mathcal{E}}_i\oplus\widetilde{\mathcal{E}}_i}, D^V_{\widetilde{\mathcal{E}}_i}]=0.$$

Proof. It suffices to show that any form ν in $\mathcal{H}^{0,1}(\Sigma,\mathrm{End}(E_\rho))$ is necessarily of rapid decay as one approaches a cusp (in the coordinates (2.6)), for then the results easily follows from (3.16) and (3.17).

To see that the forms in $\mathcal{H}^{0,1}(\Sigma,\mathrm{End}(E_\rho))$ have the claimed behavior, notice first that if ν is in $\mathcal{H}^{0,1}(\Sigma,\mathrm{End}(E_\rho))$, then $*\nu$ is in $\mathcal{H}^{1,0}(\Sigma,\mathrm{End}(E_\rho))$. In the complex coordinate $\zeta_i=e^{2\pi i\sigma_i^{-1}z}$ with $\zeta_i(p_i)=0$ near a puncture $p_i\in\overline{\Sigma}$, the holomorphic form $*\nu$ has the form

$$*\,\nu(z)=\sum_{l,m=1}^{r_i}\sum_{k=0}^{\infty}a_k^{ilm}e^{2\pi i(\alpha_l(p_i)-\alpha_m(p_i))z}e^{2\pi ikz}dz \tag{3.18}$$

where $a_k^{ilm}\in W_{il}\otimes W^*_{im}$. To insure that $*\nu$ is square integrable, we need that $a_0^{ilm}=0$ when $\alpha_l(p_i)-\alpha_m(p_i)\le 0$. In particular, this implies that in the coordinates (2.6), both $*\nu$ and $\nu=-*(*\nu)$ are of rapid decay as one approaches a puncture. □

4. The index formulæ

To compute the index formulæ of the families of operators $\overline{\partial}_E$ and $\overline{\partial}_{\mathcal{E}}$ in (3.5), we will use the results of [1]. The main step consists in computing the eta forms of the vertical families of these operators.

For $i\in\{1,\dots,n\}$ and $j\in\{1,\dots,k_i\}$, let $E_{ij}\to\mathcal{N}$ be the Hermitian vector bundle of rank $k_j(p_i)$ with fibre at $[\rho]\in\mathcal{N}$ given by the eigenspace corresponding to the eigenvalue $e^{2\pi i\alpha_j(p_i)}$ of $\sigma_{[\rho]}(S_i)$. For each i, consider the Hermitian vector bundle obtained by taking the direct sum

$$E_i:=\bigoplus_{j=1}^{r_i}E_{ij}. \tag{4.1}$$

The bundle E_i is clearly topologically trivial, but it is not necessarily trivial as a Hermitian vector bundle. By lemma 3.2, the Chern connection ∇^E

induces a connection on each of the bundles E_{ij}. This is just the Chern connection $\nabla^{E_{ij}}$ of E_{ij}. There is also an induced connection on E_i, namely its Chern connection given by

$$\nabla^{E_i} = \bigoplus_{j=1}^{r_i} \nabla^{E_{ij}} \tag{4.2}$$

Lemma 4.1. *The (renormalized) eta form of the vertical family $\overline{\partial}_E^V$ is given by*

$$\widehat{\eta}(\overline{\partial}_E^V) = \sum_{i=1}^{n} \sum_{\alpha_j(p_i)>0} \left(\frac{1}{2} - \alpha_j(p_i)\right) \mathrm{Ch}(E_{ij}),$$

where $\mathrm{Ch}(E_{ij}) := \mathrm{Tr}(\exp(\frac{i(\nabla^{E_{ij}})^2}{2\pi}))$ *is a form representing the Chern character of E_{ij}.*

Proof. Recall from [8] (we will use the notation of [30]) that the eta form of the self-adjoint family $\overline{\partial}_{E_i}^V$ is given by

$$\eta(\overline{\partial}_{E_i}^V) = \frac{1}{\sqrt{\pi}} \int_0^\infty \mathrm{STr}_{\mathbb{C}l(1)} \left(\frac{d\mathbb{B}_t}{dt} e^{-\mathbb{B}_t^2} \right) dt \tag{4.3}$$

where $\mathbb{B}_t = t^{\frac{1}{2}} \mathbb{B}_{[0]} + \mathbb{B}_{[1]} + t^{-\frac{1}{2}} \mathbb{B}_{[2]}$ is the rescaled Bismut superconnection of the family $\overline{\partial}_{E_i}^V$. It is given by

$$\begin{aligned} \mathbb{B}_{[0]} &= \sigma \overline{\partial}_{E_i}^V, \\ \mathbb{B}_{[1]} &= \sum_\alpha \varepsilon(f^\alpha)(\nabla^{\widetilde{E}_i \oplus \widetilde{E}_i} + \frac{1}{2} k^{\partial\widetilde{\pi}}(f_\alpha)), \\ \mathbb{B}_{[2]} &= \sigma \frac{1}{4} \sum_{\alpha<\beta} \varepsilon(f^\alpha)\varepsilon(f^\beta) c(g_i) \Omega^{\partial\widetilde{\pi}}(f_\alpha, f_\beta)(g_i). \end{aligned} \tag{4.4}$$

Here, $\{f_\alpha\}$ is a local orthonormal basis of the horizontal tangent space, $\{g_i\}$ is a local orthonormal basis of the vertical tangent space and σ is the matrix

$$\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Since the fibration $\partial\widetilde{\pi} : \partial\widetilde{\Sigma} \times \mathcal{N} \to \mathcal{N}$ is trivial, the curvature $\Omega^{\partial\widetilde{\pi}}$ and the second fundamental form $k^{\partial\widetilde{\pi}}$ both vanish identically. Thus, we have in fact

$$\mathbb{B}_{[0]} = \sigma \overline{\partial}_{\widetilde{E}_i}^V, \quad \mathbb{B}_{[1]} = \sum_\alpha \varepsilon(f^\alpha) \nabla^{\widetilde{E}_i \oplus \widetilde{E}_i}, \quad \mathbb{B}_{[2]} = 0. \tag{4.5}$$

By lemma 3.2, the family $\overline{\partial}_{E_i}^V$ is parallel with respect to the horizontal connection, $[\nabla^{\widetilde{E}_i \oplus \widetilde{E}_i}, \sigma \overline{\partial}_{E_i}^V] = 0$, which implies that

$$\mathbb{B}_{[0]}\mathbb{B}_{[1]} + \mathbb{B}_{[1]}\mathbb{B}_{[0]} = 0.$$

Consequently, $\mathbb{B}_t^2 = t\mathbb{B}_{[0]}^2 + \mathbb{B}_{[1]}^2 = t(\overline{\partial}_{E_i}^V)^2 + \Omega^{\widetilde{E}_i \oplus \widetilde{E}_i}$. The formula for the eta form therefore simplifies to

$$\begin{aligned}\eta(\overline{\partial}_{E_i}^V) &= \frac{1}{\sqrt{\pi}} \int_0^\infty \mathrm{STr}_{\mathbb{C}l(1)} \left(\frac{1}{2t^{\frac{1}{2}}} \sigma \overline{\partial}_{\widetilde{E}_i}^V e^{-t(\overline{\partial}_{\widetilde{E}_i}^V)^2 - \Omega_H^{\widetilde{E}_i \oplus \widetilde{E}_i}} \right) dt \\ &= \frac{1}{\sqrt{\pi}} \int_0^\infty \mathrm{Tr} \left(\frac{1}{2t^{\frac{1}{2}}} \overline{\partial}_{\widetilde{E}_i}^V e^{-t(\overline{\partial}_{\widetilde{E}_i}^V)^2 - \Omega_H^{\widetilde{E}_i}} \right) dt.\end{aligned} \tag{4.6}$$

where

$$\Omega_H^{\widetilde{E}_i} = \varepsilon(f^\alpha)\varepsilon(f^\beta)\Omega^{\widetilde{E}_i}(f_\alpha, f_\beta)$$

is the horizontal contribution to the curvature of $\widetilde{E}_i$. It is such that

$$\Omega_H^{\widetilde{E}_i} = \partial\widetilde{\pi}^* \Omega^{E_i}$$

where Ω^{E_i} is the curvature of the connection ∇^{E_i}. The fact that $\overline{\partial}_{\widetilde{E}_i}^V$ is parallel with respect to the horizontal connection also means that $\overline{\partial}_{\widetilde{E}_i}^V$ commutes with $\Omega_H^{\widetilde{E}_i}$. In terms of the decomposition

$$\overline{\partial}_{E_i}^V = \bigoplus_{j=1}^{r_i} \overline{\partial}_{E_{ij}}^V, \tag{4.7}$$

this means that the eta form is given by

$$\eta(\overline{\partial}_E^V) \sum_{j=1}^{r_i} \eta(\overline{\partial}_{E_{ij}}^V) = \sum_{j=1}^{r_i} \frac{\eta(\overline{\partial}_{E_{ij}}^V)_{[0]}}{k_j(p_i)} \mathrm{Tr}\left(e^{-\Omega^{E_{ij}}}\right) \tag{4.8}$$

where $\Omega^{E_{ij}} = (\nabla^{E_{ij}})^2$. The result then follows by using lemma 2.7 and summing over i with the **renormalized** eta form given by (cf. [8], remark 4.101)

$$\widehat{\eta}(\overline{\partial}_E^V) := \sum_k \frac{1}{(2\pi i)^k} \eta(\overline{\partial}_E^V)_{[2k]}. \tag{4.9}$$

□

Theorem 4.2. *The Chern character of the families index of $\overline{\partial}_E$ is represented in de Rham cohomology by the form*

$$\pi_*(\mathrm{Td}(\Sigma)\,\mathrm{Ch}(E)) - \sum_{i=1}^n \sum_{j=1}^{r_i} \left(\frac{1}{2} - \alpha_j(p_i)\right) \mathrm{Ch}(E_{ij}) + \sum_{\alpha_1(p_i)=0} \mathrm{Ch}(E_{i1}). \tag{4.10}$$

Proof. According to the index formula of ([1], theorem 4.5), the Chern character of the family index is represented by the form

$$\pi_*(\widehat{A}(g_\Sigma)\,\mathrm{Ch}'(E)) - \widehat{\eta}(\overline{\partial}_E^V) - \widehat{\eta}(\overline{\partial}_E^H). \tag{4.11}$$

Since $T(\Sigma \times \mathcal{N}) = T\Sigma \oplus T\mathcal{N}$ is an orthogonal decomposition with respect to the Kähler metric $g_\Sigma \oplus g_\mathcal{N}$, there is a standard identification of forms

$$\pi_*(\widehat{A}(g_\Sigma)\,\mathrm{Ch}'(E)) = \pi_*(\mathrm{Td}(\Sigma)\,\mathrm{Ch}(E)). \tag{4.12}$$

Moreover, we computed the renormalized eta form of $\overline{\partial}^V_{\widetilde{E}}$ in lemma 4.1. Thus, it remains to compute the (renormalized) eta form of the horizontal family $\overline{\partial}^H_{\widetilde{E}_i}$ for those i such that $\alpha_1(p_i) = 0$. From ([1], (4.12)), we have,

$$\widehat{\eta}(\overline{\partial}^H_{E_i}) = \frac{1}{2}\operatorname{sign}\left(-\frac{1}{2}\right)\operatorname{Ch}(\ker \overline{\partial}^V_{E_i}) = -\frac{1}{2}\operatorname{Ch}(\ker \overline{\partial}^V_{E_i}). \tag{4.13}$$

Since $\ker \overline{\partial}^V_{E_i}$ is canonically identified with E_{i1}, this gives

$$\widehat{\eta}(\overline{\partial}^H_{E_i}) = -\frac{1}{2}\operatorname{Ch}(E_{i1}). \tag{4.14}$$

Combining this with the computation of the first two terms of (4.2), the result follows. □

We have similarly an explicit formula for the Chern character of the index bundle of $\overline{\partial}_{\mathcal{E}}$, starting with the following expression for its eta form.

Lemma 4.3. *The renormalized eta form of $\overline{\partial}^V_{\widetilde{\mathcal{E}}}$, $\widehat{\eta}(\overline{\partial}^V_{\mathcal{E}})$, is given by*

$$\sum_{i=1}^{n}\sum_{j\neq l}\frac{\operatorname{sign}(\alpha_j(p_i)-\alpha_l(p_i))(1-2|\alpha_j(p_i)-\alpha_l(p_i)|)}{2}\operatorname{Ch}(E_{ij})\operatorname{Ch}(E^*_{il}).$$

Proof. As in the proof of lemma 4.1, the fibration $\partial\widetilde{\pi} : \partial\widetilde{\Sigma}\times\mathcal{N} \to \mathcal{N}$ is trivial and has curvature $\Omega^{\partial\widetilde{\pi}}$ and second fundamental form $k^{\partial\widetilde{\pi}}$ vanishing identically. By lemma 3.2, the vertical family is also parallel with respect to the connection of $\partial\widetilde{\pi}_*\widetilde{\mathcal{E}}_i$. Thus, using the decomposition

$$\overline{\partial}^V_{\widetilde{\mathcal{E}}_i} = \bigoplus_{j,l}\overline{\partial}^V_{\hom(\widetilde{E}_{il},\widetilde{E}_{jl})} \tag{4.15}$$

and proceeding as in the proof of lemma 4.1, we can write the eta form of the vertical family $\overline{\partial}^V_{\mathcal{E}_i}$ at the ith boundary component as

$$\begin{aligned}\widehat{\eta}(\overline{\partial}^V_{\mathcal{E}_i}) &= \widehat{\eta}\left(\bigoplus_{j,l}\overline{\partial}^V_{\hom(\widetilde{E}_{il},\widetilde{E}_{ij})}\right),\\ &= \sum_{j,l}\frac{\eta(\overline{\partial}_{\hom(E_{il},E_{ij})})_{[0]}}{k_j(p_i)k_l(p_i)}\operatorname{Ch}(\hom(E_{il},E_{ij})).\end{aligned} \tag{4.16}$$

Since $\hom(E_{il},E_{ij}) = E_{ij}\otimes E^*_{il}$ and $\eta(\overline{\partial}_{\hom(\widetilde{E}_{il},\widetilde{E}_{ij})})_{[0]}$ is $\frac{1}{2}$ the eta invariant of the family, we conclude from lemma 2.7 that

$$\widehat{\eta}(\overline{\partial}^V_{\widetilde{\mathcal{E}}_i}) = \sum_{j\neq l}\frac{\operatorname{sign}(\alpha_j(p_i)-\alpha_l(p_i))(1-2|\alpha_j(p_i)-\alpha_l(p_i)|)}{2}\operatorname{Ch}(E_{ij})\operatorname{Ch}(E^*_{il}). \tag{4.17}$$

Summing over i to get the contributions from all the parabolic points, we get the result. □

For a fixed fibre of the moduli space $\mathcal{N}$, the identity section $\mathrm{Id}_{E_\rho} \in \mathrm{End}(E_\rho)$ is obviously in the kernel of $\overline{\partial}_{\mathcal{E}_\rho}$. In fact, up to a constant, this is the only element, since the eigenspaces of any other element of the kernel would decompose E into parabolic subbundles, contradicting the stability of E_ρ. In particular, the dimension of the kernel does not jump as one moves on the moduli space and there is a well-defined kernel bundle $\ker \overline{\partial}_{\mathcal{E}}$ trivialized by the identity section $\mathrm{Id}_{\mathcal{E}} \in \mathrm{End}(E)$. Consequently, there is also a cokernel bundle $\ker \overline{\partial}^*_{\mathcal{E}} \to \mathcal{N}$. As above, the connection of $\pi_*\mathcal{E}$ induces connections on $\ker \overline{\partial}_{\mathcal{E}}$ and $\ker \overline{\partial}^*_{\mathcal{E}}$. With respect to the trivialization of the identity section $\mathrm{Id}_{\mathcal{E}}$, this connection corresponds to the trivial connection. Thus, in this context, the Chern character of the index bundle of the family $\overline{\partial}_{\mathcal{E}}$ is represented by the form

$$\mathrm{Ch}(\ker \overline{\partial}_{\mathcal{E}}, \nabla^{\ker \overline{\partial}_{\mathcal{E}}}) - \mathrm{Ch}(\ker \overline{\partial}^*_{\mathcal{E}}, \nabla^{\ker \overline{\partial}^*_{\mathcal{E}}}) = 1 - \mathrm{Ch}(\ker \overline{\partial}^*_{\mathcal{E}}, \nabla^{\ker \overline{\partial}^*_{\mathcal{E}}}) \tag{4.18}$$

Theorem 4.4. *At the level of forms, the Chern character of the index bundle of $\overline{\partial}_{\mathcal{E}}$ is given by*

$$\begin{aligned} 1 - \mathrm{Ch}(\ker \overline{\partial}^*_{\mathcal{E}}, \nabla^{\ker \overline{\partial}^*_{\mathcal{E}}}) &= \pi_*(\mathrm{Td}(\Sigma)\,\mathrm{Ch}(\mathcal{E})) - \sum_{i=1}^{n} \sum_{j\neq l} \mu^i_{jl}\,\mathrm{Ch}(E_{ij})\,\mathrm{Ch}(E^*_{il}) \\ &\quad + \frac{1}{2}\sum_{i=1}^{n}\sum_{l=1}^{r_i} \mathrm{Ch}(E_{il})\,\mathrm{Ch}(E^*_{il}) \\ &\quad - \left(\frac{1}{2\pi i}\right)^{\frac{N}{2}} d\int_0^\infty \mathrm{Str}\left(\mathbb{A}^t_{D_{\mathcal{E}}} e^{-(\mathbb{A}^t_{D_{\mathcal{E}}})^2}\right) dt \end{aligned}$$

where $\mathbb{A}^t_{D_{\mathcal{E}}}$ is the rescaled Bismut superconnection of $D_{\mathcal{E}} = \sqrt{2}(\overline{\partial}_{\mathcal{E}} + \overline{\partial}^*_{\mathcal{E}})$ and

$$\mu^i_{jl} := \mathrm{sign}(\alpha_j(p_i) - \alpha_l(p_i))\left(\frac{1}{2} - |\alpha_j(p_i) - \alpha_l(p_i)|\right)$$

Proof. According to ([1], Theorem 4.5), we have the following formula,

$$\begin{aligned} 1 - \mathrm{Ch}(\ker \overline{\partial}^*_{\mathcal{E}}, \nabla^{\ker \overline{\partial}^*_{\mathcal{E}}}) &= \pi_*(\widehat{A}(\Sigma)\,\mathrm{Ch}'(\mathcal{E})) - \widehat{\eta}(\overline{\partial}^V_{\mathcal{E}}) - \widehat{\eta}(\overline{\partial}^H_{\mathcal{E}}) \\ &\quad - \left(\frac{1}{2\pi i}\right)^{\frac{N}{2}} d\int_0^\infty \mathrm{Str}\left(\mathbb{A}^t_{D_{\mathcal{E}}} e^{-(\mathbb{A}^t_{D_{\mathcal{E}}})^2}\right) dt. \end{aligned} \tag{4.19}$$

As in theorem 4.2, the first term on the right hand side can be rewritten in terms of the Todd form,

$$\pi_*(\widehat{A}(\Sigma)\,\mathrm{Ch}'(\mathcal{E})) = \pi_*(\mathrm{Td}(\Sigma)\,\mathrm{Ch}(E)).$$

The second term was computed in lemma 4.3. For the third term, we have according to formula 4.12 in [1],

$$\begin{aligned} \widehat{\eta}(\overline{\partial}^H_{\widetilde{\mathcal{E}}_i}) &= \tfrac{1}{2}\,\mathrm{sign}\left(-\tfrac{1}{2}\right)\mathrm{Ch}(\ker \overline{\partial}^V_{\widetilde{\mathcal{E}}_i}) \\ &= -\tfrac{1}{2}\textstyle\sum_{j=1}^{r_i} \mathrm{Ch}(E_{ij})\,\mathrm{Ch}(E^*_{ij}), \end{aligned} \tag{4.20}$$

since $\ker \overline{\partial}^V_{\widetilde{\mathcal{E}}_i}$ is canonically identified with $\sum_{j=1}^{r_i} E_{ij} \otimes E_{ij}^*$. Summing over i, we get

$$\widehat{\eta}(\overline{\partial}^H_{\mathcal{E}}) = -\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{r_i} \mathrm{Ch}(E_{ij}) \, \mathrm{Ch}(E_{ij}^*). \tag{4.21}$$

□

5. The curvature of the determinant line bundle

In general, the geometry encoded in the Chern character of the index bundle is hard to unravel. The exception is the two-form part of the Chern character which is known to be the curvature of the determinant line bundle. This is true at the level of forms if these bundles are endowed with, respectively, the Bismut superconnection and the Quillen metric and connection (whose definition we now recall).

The determinant line bundle of a holomorphic family of Fredholm $\overline{\partial}$ operators D_z is a line bundle over the parameter space, which at every point satisfies

$$(\mathrm{Det}\, D)_z \cong (\Lambda^{\max} \ker D_z) \otimes (\Lambda^{\max} \operatorname{coker} D_z)^*. \tag{5.1}$$

If the null spaces of D_z fit together to form a vector bundle, then the right hand side of (5.1) serves as the definition of the determinant bundle. This is the case for instance for the family $D_{\mathcal{E}}$. If the the null spaces of D_z do not form a bundle (e.g., when the dimension varies with z), but the spectrum of each D_z is entirely made up of eigenvalues of finite multiplicity, then $\mathrm{Det}\, D$ can be constructed by a truncation procedure from [35], [9] (see also [6]). This is the case when the operators D_z act on compact spaces; it is also the case for the family D_E when all of the parabolic weights are non-zero by results of [39], as explained above.

If the operators D_z act on sections of a holomorphic bundle over a closed manifold, the line bundle $\mathrm{Det}\, D$ has a canonical choice of metric and connection defined using the zeta-regularized determinant of the family $D_z^* D_z$. Recall that the zeta function of $D_z^* D_z$ is defined, for $\xi \gg 0$, by

$$\zeta_{D_z^* D_z}(\xi) = \frac{1}{\Gamma(\xi)} \int_0^\infty t^\xi \, \mathrm{Tr}(e^{-tD_z^* D_z} - \mathcal{P}_{\ker D_z^* D_z}) \frac{dt}{t}.$$

The short-time asymptotics of the heat kernel allow this function to be meromorphically continued to the whole complex plane. The origin is a regular point of the extension and the derivative at the origin is used to define the determinant of $D_z^* D_z$ by

$$\log \det D_z^* D_z = -\zeta'_{D_z^* D_z}(0).$$

When $\ker D$ and $\operatorname{coker} D$ form actual bundles, the Quillen metric on the determinant line bundle is defined by starting with the L^2 metric induced

from (5.1), $\|\cdot\|$, and then adjusting by the zeta-regularized determinant of $D_z^* D_z$,

$$\|\cdot\|_Q = (\det D_z^* D_z)^{-1/2} \|\cdot\|. \tag{5.2}$$

When the nullspaces of D_z do not form a bundle, the determinant line bundle does not have a well-defined induced L^2-metric, but the Quillen metric does have a natural generalization which is globally well-defined. The Quillen metric is a Hermitian metric on the holomorphic line bundle $\operatorname{Det} D$ and hence has a unique compatible connection, the Chern connection. It is well-known that the curvature of this connection coincides with the two-form part of the Chern character of the Bismut superconnection on the index bundle of the family D_z.

If the prescribed weights are all non-zero, we have seen in proposition 2.3 that the operators of the family D_E all have discrete spectrum. In fact, by the explicit construction of the heat kernel given in [39], we also know that

$$e^{-tD^2_{E_\rho}} \in \rho_\Sigma^\infty \Psi_b^{-\infty}(\widetilde{\Sigma}; (\underline{\mathbb{C}} \oplus {}^{\mathrm{hc}}\Lambda^{0,1}_\Sigma) \otimes \widetilde{E}_\rho) \tag{5.3}$$

for $t > 0$ and $[\rho] \in \mathcal{N}$ (Ψ_b denotes the space of b-pseudodifferential operators, see [27]). In the family case, a similar statement holds for the heat kernel of the rescaled Bismut superconnection $\mathbb{A}_t^E$. In this non-compact context, the class of operators $\rho^\infty \Psi_b^{-\infty}(\widetilde{\Sigma})$ is really the analog of smoothing operators on a compact manifold. For instance, these operators are of trace class (while general operators in $\Psi_b^{-\infty}(\widetilde{\Sigma})$ are not).

The fact the spectrum of D_{E_ρ} is discrete and its heat kernel satisfies (5.3) indicates that the family D_E spectrally behaves as a family of Dirac type operators on a compact manifold. Because of this, the standard definition of the Quillen metric and connection and the computation of its curvature for families of Dirac type operators acting on compact manifolds (as in [35], [9] or [6]) generalize almost immediatly to the family D_E. The only difference is that (see corollary A.4 in Appendix A) there are potentially extra powers of $\sqrt{t}$ involved in the asymptotic expansion of the trace of the heat kernel,

$$\operatorname{Tr}(e^{-tD^2_{E_\rho}}) = \frac{a_{-1}}{t} + \frac{a_{-\frac{1}{2}}}{\sqrt{t}} + a_0 + \mathcal{O}(\sqrt{t}) \quad \text{as } t \to 0^+. \tag{5.4}$$

But the discussion in, for instance, [6] works equally well with these extra asymptotic terms. This gives the following.

Theorem 5.1. *When the weights of the parabolic structure are all non-zero, the curvature of the Quillen connection ∇^{Q_E} associated to the determinant line of the family of operators D_E is given by*

$$\frac{i}{2\pi}(\nabla^{Q_E})^2 = \pi_*(\operatorname{Td}(\Sigma)\operatorname{Ch}(E))_{[2]} - \sum_{i=1}^{n}\sum_{j=1}^{r_i}(\frac{1}{2} - \alpha_j(p_i))c_1(E_{ij})$$

One can also consider, instead of the family $\overline{\partial}_E$, the corresponding family $\overline{\partial}_{\overline{E}}$ on the compactified fibration $\overline{\Sigma} \times \mathcal{N} \to \mathcal{N}$. In principle, this leads to a different determinant line bundle since according to theorem 4.2, $\overline{\partial}_{E_\rho}$ and $\overline{\partial}_{\overline{E}_\rho}$ have in general different indices. When the weights are non-zero and rational, Biswas and Raghavendra, in [10], computed the curvature of the determinant line bundle of the family $\overline{\partial}_{\overline{E}}$ defined on the fibration $\overline{\Sigma} \times \mathcal{N} \to \mathcal{N}$. Their approach consisted in 'unfolding' the parabolic structure by lifting the family of operators $\overline{\partial}_{\overline{E}}$ to an appropriate cover $Y \to \overline{\Sigma}$ ramified at the marked points $p_1, \ldots, p_n$. On this ramified cover Y, they could use the idea of Quillen [35] to compute the curvature of the Quillen connection and relate it to the natural symplectic form of the moduli space.

For the family of operators $\overline{\partial}_{\mathcal{E}}$, there is also no problem defining the determinant bundle since the kernel and cokernel of $\overline{\partial}_{\mathcal{E}}$ form vector bundles and we can define $\operatorname{Det} \overline{\partial}_{\mathcal{E}}$ directly by (5.1). Moreover, since the bundle $\ker \overline{\partial}_{\mathcal{E}}$ is trivial, we have

$$(\operatorname{Det} \overline{\partial}_{\mathcal{E}})_z = (\Lambda^{\max} \operatorname{coker}(\overline{\partial}_{\mathcal{E}})_z)^*.$$

The definition of the Quillen metric on $\operatorname{Det} \overline{\partial}_{\mathcal{E}}$ is not as straightforward since their heat kernels need not be of trace class. Indeed, recall that if $\mathcal{K}_t(\zeta, \zeta')$ denotes the integral kernel of the heat kernel of $D_z^* D_z$, so that

$$e^{-tD_z^* D_z} f = \int \mathcal{K}_t(\zeta, \zeta') f(\zeta') \, d\zeta',$$

then Lidskii's theorem says that the trace of $e^{-tD_z^* D_z}$ when it exists is given by

$$\operatorname{Tr} e^{-tD_z^* D_z} = \int \mathcal{K}_t(\zeta, \zeta) \, d\zeta.$$

On a non-compact manifold, $\mathcal{K}_t(\zeta, \zeta)$ will be a smooth function but need not be integrable. This is the case for $\overline{\partial}_{\mathcal{E}}^* \overline{\partial}_{\mathcal{E}}$ for which we have a very precise description of the heat kernel from Vaillant's thesis [39, §4]. Nevertheless, from this description we know that, if x is a boundary defining function and $\varepsilon > 0$, then $\int_{x \geq \varepsilon} \mathcal{K}_t(\zeta, \zeta)$ is finite and has an asymptotic expansion in ε so we can define

$$^R\operatorname{Tr}\left(e^{-tD_z^* D_z}\right) = {}^R\!\!\int \mathcal{K}_t(\zeta, \zeta) \, d\zeta = \operatorname*{FP}_{\varepsilon=0} \int_{x \geq \varepsilon} \mathcal{K}_t(\zeta, \zeta) \, d\zeta.$$

This is a functional that coincides with the trace on operators of trace class, but that does not necessarily vanish on commutators.

The renormalized trace extends to $e^{-tD_z^* D_z} - \mathcal{P}_{\ker D_z^* D_z}$ and so we define

$$^R\zeta_{\overline{\partial}_{\mathcal{E}}^* \overline{\partial}_{\mathcal{E}}}(\xi) = \frac{1}{\Gamma(\xi)} \int_0^\infty t^\xi \, {}^R\operatorname{Tr}(e^{-t\overline{\partial}_{\mathcal{E}}^* \overline{\partial}_{\mathcal{E}}} - \mathcal{P}_{\ker \overline{\partial}_{\mathcal{E}}^* \overline{\partial}_{\mathcal{E}}}) \frac{dt}{t}.$$

The description of the heat kernel in Vaillant's thesis also implies an expansion in t as $t \to 0^+$ (see appendix A below) which allows us to extend $^R\zeta_{\overline{\partial}_{\mathcal{E}}^* \overline{\partial}_{\mathcal{E}}}$

meromorphically to the whole complex plane and define

$$\log\det \overline{\partial}^*_{\mathcal{E}}\overline{\partial}_{\mathcal{E}} = -{}^R\zeta'_{D_z^*D_z}(0) \tag{5.5}$$

and then define $\|\cdot\|_Q$ by (5.2).

To define the Quillen connection of the determinant line bundle $\det\overline{\partial}_{\mathcal{E}}$, we can take the Chern connection with respect to its Quillen metric. However, to compute its curvature, it is better to use an alternative definition in terms of heat kernels. This require some preparation. On the moduli space $\mathcal{N}$, we need to consider the Fréchet bundle $\pi_*\mathcal{E}\to\mathcal{N}$ whose fibre at $[\rho]\in\mathcal{N}$ is given by

$$\pi_*\mathcal{E}_{[\rho]} = \dot{\mathcal{C}}^\infty(\Sigma;\mathcal{E}_{\sigma([\rho])}\otimes(\Lambda^{0,0}_\Sigma\oplus\Lambda^{0,1}_\Sigma)\otimes|{}^{\mathrm{hc}}\Lambda_\Sigma|^{\frac12}) \tag{5.6}$$

where $|{}^{\mathrm{hc}}\Lambda_\Sigma|^{\frac12}$ is the half-density bundle on Σ and the symbol $\dot{\mathcal{C}}^\infty$ means we consider smooth sections with rapid decay as one approaches a puncture. Since the bundle $|{}^{\mathrm{hc}}\Lambda_\Sigma|$ is canonically trivialized by the volume form of the hyperbolic metric g_Σ, the families of Dirac type operators

$$D_{\mathcal{E}} := \sqrt{2}(\overline{\partial}_{\mathcal{E}}+\overline{\partial}^*_{\mathcal{E}}),\quad D^+_{\mathcal{E}} = \sqrt{2}\overline{\partial}_{\mathcal{E}},\quad D^-_{\mathcal{E}} = \sqrt{2}\overline{\partial}^*_{\mathcal{E}} \tag{5.7}$$

naturally act on sections of $\pi_*\mathcal{E}_{[\rho]}$. The connection $\nabla^{\mathcal{E}}$ described in (3.12) then naturally induces a covariant derivative $\nabla^{\pi_*\mathcal{E}}$ on the Fréchet bundle $\pi_*\mathcal{E}\to\mathcal{N}$. This allows one to define the rescaled superconnection

$$\mathbb{A}^s_{\mathcal{E}} := s^{\frac12}D_{\mathcal{E}} + \nabla^{\pi_*\mathcal{E}} \tag{5.8}$$

and for $s\in\mathbb{R}^+$ the differential forms

$$\alpha^\pm(s) := {}^R\operatorname{Tr}_{\pi_*\mathcal{E}^\pm}\left(\frac{\partial\mathbb{A}^s_{\mathcal{E}}}{\partial s}e^{-(\mathbb{A}^s_{\mathcal{E}})^2}\right). \tag{5.9}$$

As in equation (8.12) of [2], the integrals

$$\beta^\pm_{\mathcal{E}}(z) := \int_0^\infty t^z\alpha^\pm_{\mathcal{E}}(t)_{[1]}dt \tag{5.10}$$

are holomorphic for $\operatorname{Re} z\gg 0$ and admit meromorphic extensions to the whole complex plane. In particular, one can consider the 1-forms

$$\beta^\pm_{\mathcal{E}} := \operatorname*{FP}_{z=0}\frac{d}{dz}\left(\frac{1}{\Gamma(z)}\beta^\pm_{\mathcal{E}}(z)\right). \tag{5.11}$$

The orthogonal projections $P_\pm:\pi_*\mathcal{E}^\pm\to\ker D^\pm_{\mathcal{E}}$ induce connections

$$\nabla^{\ker D^\pm_{\mathcal{E}}} := P_\pm\nabla^{\pi_*\mathcal{E}^\pm}P_\pm, \tag{5.12}$$

and so a connection $\nabla^{\det\overline{\partial}_{\mathcal{E}}}$ on the determinant line bundle $\det\overline{\partial}_{\mathcal{E}}$. This connection is holomorphic and is the Chern connection with respect to the L^2-metric induced on $\det\overline{\partial}_{\mathcal{E}}$. The **Quillen connection** on $\det\overline{\partial}_{\mathcal{E}}$ is defined to be the connection

$$\nabla^{Q_{\mathcal{E}}} := \nabla^{\det\overline{\partial}_{\mathcal{E}}} + \beta^+_{\mathcal{E}}. \tag{5.13}$$

Proposition 5.2. *The Quillen connection is the Chern connection of* $\det \overline{\partial}_{\mathcal{E}}$ *with respect to the Quillen metric* $\| \cdot \|_{Q_{\mathcal{E}}}$.

The proof of this proposition is essentially the same as the one of proposition 8.4 in [2]. We will therefore not repeat it here, but simply point out that in [2], lemma 8.1 was the key fact that allowed one to proceed as in the compact case (cf. [6] and [9]). In our context, the equivalent of this fact is the following.

Lemma 5.3. *The Schwartz kernel of* $[\nabla^{\pi_*\mathcal{E}}, D^{\pm}_{\mathcal{E}}]$ *vanishes to all order at the front face. In particular, for* $P \in \Psi^{-\infty}(\Sigma \times \mathcal{N}/\mathcal{N}; \mathcal{E} \otimes (\Lambda^{0,0}_{\Sigma} \oplus \Lambda^{0,1}_{\Sigma}) \otimes |\Lambda_{\Sigma}|^{\frac{1}{2}})$,

$$^{R}\operatorname{STr}\left([[\nabla^{\pi_*\mathcal{E}}, D^{\pm}_{\mathcal{E}}], P]\right) = 0.$$

Proof. The fact that $[\nabla^{\pi_*\mathcal{E}}, D^{\pm}_{\mathcal{E}}]$ vanishes to all order at the front face follows from (3.16) and the fact that a form ν in $\mathcal{H}^{0,1}(\Sigma, \operatorname{End}(E_{\sigma([\rho])}))$ is necessarily of rapid decay as one approaches a puncture (see the proof of lemma 3.2). Now, it is well-known that

$$^{R}\operatorname{STr}\left([[\nabla^{\pi_*\mathcal{E}}, D^{\pm}_{\mathcal{E}}], P]\right)$$

only depends linearly on the asymptotic expansion of the Schwartz kernels of P and $[\nabla^{\pi_*\mathcal{E}}, D^{\pm}_{\mathcal{E}}]$ at the corner of $\widetilde{\Sigma} \times \widetilde{\Sigma}$. The asymptotic expansion of $[\nabla^{\pi_*\mathcal{E}}, D^{\pm}_{\mathcal{E}}]$ being trivial, the result follows. □

With this lemma, the proof of proposition 5.2 is essentially the same as the one of proposition 8.1 in [2]. We refer to [2] for further details. We can now proceed as in the closed case (cf. [9] and [6], see also [34] for the b-case) to compute the curvature of the Quillen connection.

Theorem 5.4. *The curvature of the Quillen connection on the determinant line bundle of* $\overline{\partial}_{\mathcal{E}}$ *is given by*

$$\begin{aligned}\frac{i}{2\pi}(\nabla^{Q_{\mathcal{E}}})^2 &= \pi_*(\operatorname{Td}(\Sigma)\operatorname{Ch}(\mathcal{E}))_{[2]} \\ &\quad - \sum_{i=1}^{n}\sum_{j\neq l} \operatorname{sign}(\alpha_j(p_i) - \alpha_l(p_i))(1 - 2|\alpha_j(p_i) - \alpha_l(p_i)|)k_l(p_i)c_1(E_{ij}).\end{aligned}$$

where $c_1(E_{ij}) := \frac{i}{2\pi}\operatorname{Tr}((\nabla^{E_{ij}})^2)$ *is the first Chern form of* E_{ij} *with respect to its naturally induced connection* $\nabla^{E_{ij}}$.

Proof. As in the case where the fibres are closed manifolds and as in theorem 8.5 of [2], the curvature of the Quillen connection is simply the two form part of the Chern character of the index bundle given by theorem 4.4, except for the exact term,

$$-\left(\frac{1}{2\pi i}\right)^{\frac{N}{2}} d\int_0^{\infty} \operatorname{Str}\left(\mathbb{A}^t_{D_{\mathcal{E}}} e^{-(\mathbb{A}^t_{D_{\mathcal{E}}})^2}\right) dt$$

which does not contribute. Putting all the terms involving $c_1(E_{ij})$ together for $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, r_i\}$, we get the desired result. $\square$

Since $c_1(E_{ij}) = c_1(\det E_{ij})$ is defined at the level of form by using the Chern connection of E_{ij} (cf. lemma 4 in [38]), our formula is the same as the one recently obtained by Takhtajan and Zograf in ([38], theorem 2)[1]. In [38], Takhtajan and Zograf also identified the term $\pi_*(\mathrm{Ch}(\mathcal{E}))_{[2]}$ with the natural $(1,1)$-form of the moduli space $\mathcal{N}$,

$$\pi_*(\mathrm{Ch}(\mathcal{E}))_{[2]} = -\frac{1}{2\pi}\widetilde{\Omega}, \tag{5.14}$$

where

$$\widetilde{\Omega}\left(\frac{\partial}{\partial\varepsilon(\mu)}, \frac{\partial}{\partial\varepsilon(\nu)}\right) = \frac{i}{2}\int_\Sigma \mathrm{Tr}\left(\mathrm{ad}\,\mu \wedge \mathrm{ad}\,{*\nu}\right). \tag{5.15}$$

This allowed them to use their formula to give a new way of computing the symplectic volume form of the moduli space $\mathcal{N}$ in some special cases.

The fact we get the same formula as in [38] is certainly expected at the cohomological level, but is not so trivial at the level of forms. This is because *a priori*, a different definition of the Quillen metric and Quillen connection is used in [38]. Namely, the regularized determinant (5.5) is replaced by

$$\det{}_{TZ}(\Delta_{E_\rho}) := \left.\frac{\partial}{\partial s}\right|_{s=1} Z(s, \Gamma; \mathrm{Ad}\,\rho) \tag{5.16}$$

where $Z(s, \Gamma; \mathrm{Ad}\,\rho)$ is the Selberg Zeta function associated to the operator Δ_E (see [38] for more details and references).

The fact we get the same formula as in [38] implies to the following.

Corollary 5.5. *Suppose that for a fixed weight system the moduli space $\mathcal{N}$ is compact and admits a universal parabolic stable vector bundle. Then there is a universal constant $c_{\mathcal{N}} > 0$ such that*

$$\det(D_E^- D_E^+) = e^{-{}^R\zeta'_{D_E^- D_E^+}(0)} = c_{\mathcal{N}} \left.\frac{\partial}{\partial s}\right|_{s=1} Z(s, \Gamma; \mathrm{Ad}\,\rho). \tag{5.17}$$

Proof. Denote by $\|\cdot\|_{TZ}$ the Quillen metric used in [38]. Then there exists a smooth positive function $f : \mathcal{N} \to \mathbb{R}$ such that

$$\|\cdot\|_{TZ}^2 = f\|\cdot\|_{Q_{\mathcal{E}}}^2. \tag{5.18}$$

Since theorem 5.4 leads to the same formula as in theorem 2 of [38], we have also that

$$(\nabla^{Q_{\mathcal{E}}})^2 = (\nabla^{TZ})^2 \tag{5.19}$$

[1]There is a typographical error in the statement of theorem 2 of [38]: the sum should be only for $l \neq m$.

where ∇^{TZ} is the Chern connection associated to the Hermitian metric $\|\cdot\|_{TZ}$. Now, recall that if $s : \mathcal{U} \to \det(\overline{\partial}_{\mathcal{E}})$ is a local holomorphic section of $\det(\overline{\partial}_{\mathcal{E}})$, then the curvature of the Chern connections can be written as

$$(\nabla^{TZ})^2 = \overline{\partial}\partial \log \|s\|^2_{TZ}, \quad (\nabla^{Q_{\mathcal{E}}})^2 = \overline{\partial}\partial \log \|s\|^2_{Q_{\mathcal{E}}}. \tag{5.20}$$

Thus, combining (5.18), (5.19) and (5.20), we get

$$\overline{\partial}\partial \log f = 0. \tag{5.21}$$

Since we assume that $\mathcal{N}$ is compact and since $\mathcal{N}$ is connected (see for instance proposition 2.8 in [33]), we conclude by the maximum principle that $\log f$ and f are constant and the result follows with $c_{\mathcal{N}} = f$.

□

Remark 5.6. *As shown in* [24]*, see also* [11]*, for a generic weight system, semi-stability implies stability, which means in that case that the moduli space $\mathcal{N}$ of stable parabolic vector bundles is compact. Thus, corollary 5.5 can be reformulated as saying that* (5.17) *holds for a generic weight system.*

Of course, in the spirit of [36] (see also [15], [16], [31], [13], [18] and [2] for various generalizations in non-compact situations), Corollary 5.5 should be a direct consequence of a more general result of the form

$$\det(D_E^- D_E^+ + s(1-s)) = Z(s, \Gamma, \operatorname{Ad}\rho) G(s) \tag{5.22}$$

for some universal meromorphic function $G(s)$ depending only on g, n and the set of weights and multiplicties. Presumably, the methods of [2], where similar regularized traces are used, could be adapted to this context to get a formula of the form (5.22). In particular, one expects Corollary 5.5 to also hold when the moduli space is not compact.

Appendix A. Short time expansion of the trace of the heat kernel

The presence of extra powers of $\sqrt{t}$ in the short-time expansion of the heat kernel in (5.4) is easily explained using Vaillant's description of the heat kernel. As it is no harder, and perhaps more interesting, we explain these asymptotics in the more general context of metrics with fibered hyperbolic cusps (or ϕ-hc metrics) from [39] and [1]. Recall that a Riemannian metric g on the interior of a manifold with boundary M is a (product-type[2]) ϕ-hc metric if:

1) The boundary is the total space of a fibration

$$Z - \partial M \xrightarrow{\phi} Y$$

2) There is a collar neighborhood of the boundary $\mathrm{Coll}(\partial M)$ such that for some choice of extension of ϕ to $\widetilde{\phi} : \mathrm{Coll} \to Y$

[2]All of our considerations extend to the class of *exact* ϕ-hc metrics.

and a choice of connection for $\widetilde{\phi}$, g is a submersion metric for $\widetilde{\phi}$ of the form

$$g\big|_{\mathrm{Coll}(\partial M)} = \frac{dx^2}{x^2} + \phi^* g_Y + x^2 g_Z$$

where x is a boundary defining function, g_Y is a metric on Y, and g_Z restricts to a metric on each fiber.

We will denote the dimensions of M and Y by n and h respectively.

Recall that for a differential operator D, its heat kernel e^{-tD} is the solution to the equation

$$\begin{cases} (\partial_t + D)e^{-tD} = 0 \\ \lim_{t\to 0} e^{-tD} = \mathrm{Id} \end{cases}$$

The heat kernel acts by means of its Schwartz kernel, $\mathcal{K}$, so that

$$e^{-tD} f(\zeta) = \int \mathcal{K}(\zeta, \zeta', t) f(\zeta')\, d\zeta'$$

and, when e^{-tD} is trace-class, Lidskii's theorem gives its trace as

$$\mathrm{Tr}(e^{-tD}) = \int \mathcal{K}\big|_{\mathrm{diag}}.$$

Let $\eth$ be a Dirac-type operator associated to a ϕ-hc-metric and assume that the null space of its vertical family, $\eth^V := x\eth|_{\partial M}$, forms a bundle over Y (we say that $\eth_{\phi\text{-hc}}$ satisfies the constant rank assumption). For such an operator, Vaillant found a very precise description of the Schwartz kernel of $e^{-t\eth^2}$ as a smooth function on the interior of a manifold with corners, $HM_{\phi\text{-hc}}$ with asymptotic expansions at each of the boundary faces. This construction is carried out in [39, Chapter 4] (see also [1]).

For understanding the trace of the heat kernel, it is enough for us to recall what Vaillant's construction says about the restriction of the Schwartz kernel to the diagonal. The diagonal in M^2 pulled back to $M^2 \times \mathbb{R}^+$ can be identified with a submanifold of the interior of $HM_{\phi\text{-hc}}$, whose closure we denote diag_H. It is easy to describe diag_H directly without having to review the construction of $HM_{\phi\text{-hc}}$. It is convenient to use $\sqrt{t}$ instead of t since generally the heat kernel is smooth as a function of the former but not the latter – for instance the Euclidean heat kernel is given by $(4\pi t)^{-n/2}\exp(-|x-y|^2/4t)$. Thus we start with $M \times \mathbb{R}^+_s$, where $s = \sqrt{t}$, and introduce polar coordinates around the corner $\partial M \times \{0\}$. We can do this geometrically by radially blowing-up $\partial M \times \{0\}$, that is, we replace this submanifold with its inward-pointing unit normal bundle. The resulting manifold is denoted

$$\mathrm{diag}_H = [M \times \mathbb{R}^+_s; \partial M \times \{0\}]$$

and has three boundary faces: $\mathfrak{B}_{10}$ coming from the lift of $\partial M \times \mathbb{R}^+$ to diag_H, $\mathfrak{B}_{01}$ coming from the lift of $M \times \{0\}$ to diag_H, and the front face $\mathfrak{B}_{11}$ coming from the blow-up of $\partial M \times \{0\}$. We will use ρ_{10}, ρ_{11} and

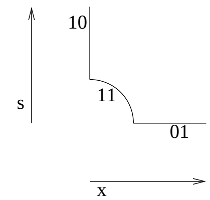

FIGURE 1. diag_H

ρ_{01} to denote boundary defining functions for these faces. From [39, Lemma 5.26] we know that the Schwartz kernel of $e^{-t\eth^2}$ satisfies

$$\mathcal{K}\big|_{\mathrm{diag}_H} \in \rho_{10}^{-1}\rho_{11}^{-h}\rho_{01}^{-n+1}\mathcal{C}^\infty(\mathrm{diag}_H, \Omega(\mathrm{diag}_H)). \tag{A.1}$$

If the vertical family $\eth^V$ is invertible, we have in fact

$$\mathcal{K}\big|_{\mathrm{diag}_H} \in \rho_{10}^{\infty}\rho_{11}^{-h}\rho_{01}^{-n+1}\mathcal{C}^\infty(\mathrm{diag}_H, \Omega(\mathrm{diag}_H)) \tag{A.2}$$

so that $e^{-t\eth^2}$ is trace class for positive time. However, when the vertical family is not invertible, only (A.1) holds and if $\mathcal{K}_t$ is the restriction of $\mathcal{K}$ to a fixed $t > 0$, then $\mathcal{K}_t\big|_{\mathrm{diag}_H}$ is not integrable and hence $e^{-t\eth^2}$ is not of trace class.

One way to define the 'trace' of the heat kernel in this case is to consider the function

$$z \mapsto \mathrm{Tr}(x^z e^{-t\eth^2}) = \int_M x^z \mathcal{K}_t\big|_{\mathrm{diag}}.$$

For $\mathrm{Re}\, z$ large enough this is well-defined and for any $t > 0$ it extends to be a meromorphic function of z with at most simple poles. We define the renormalized trace of the heat kernel to be the finite part of this function at $z = 0$,

$$^R\,\mathrm{Tr}(e^{-t\eth^2}) = {}^R\!\!\int_M \mathcal{K}_t\big|_{\mathrm{diag}} = \underset{z=0}{\mathrm{FP}} \int_M x^z \mathcal{K}_t\big|_{\mathrm{diag}}.$$

As a function of t, $\mathcal{T} = {}^R\,\mathrm{Tr}(e^{-t\eth^2})$ inherits an asymptotic expansion as $t \to 0$ from the asymptotic expansions of $\mathcal{K}$ at the boundary faces of diag_H. Each term in the expansion of $\mathcal{K}$ at either $\mathfrak{B}_{01}$ or $\mathfrak{B}_{11}$ gives rise to a term in the expansion of $\mathcal{T}$ as $t \to 0$. On the other hand, terms in the expansion of $\mathcal{K}$ at the corners $\mathfrak{B}_{01} \cap \mathfrak{B}_{11}$ and $\mathfrak{B}_{10} \cap \mathfrak{B}_{11}$ potentially give rise to extra logarithmic terms in the expansion of $\mathcal{T}$ as $t \to 0$. More precisely, we have the following.

Theorem A.1. *Given a Dirac-type operator $\eth$ associated to a ϕ-*hc*-metric with vertical family $\eth^V$ satisfying the constant rank assumption, there exist constants a_k, b_k for $k \in \mathbb{N}_0$ such that*

$$^R\,\mathrm{Tr}(e^{-t\eth^2}) \sim \left[t^{-n/2}\sum_{k\geq 0} a_k t^{k/2} + t^{-(h+1)/2}\sum_{k\geq 0} b_k t^{\frac{k}{2}}\log t \right]\ dt \tag{A.3}$$

as $t \to 0$.

Proof. Away from the corners, it is easy to see from (A.1) that the asymptotic expansions $\mathcal{K}|_{\mathrm{diag}_H}$ at the faces B_{11} and B_{01} lead to asymptotic terms of the form

$$c_k t^{\frac{-n+k}{2}}, \quad c_k \in \mathbb{C}, \ , k \in \mathbb{N}_0,$$

in the small time asymptotic expansion of ${}^R\operatorname{Tr}(e^{-t\eth^2})$. To understand how the logarithmic terms occur in the short time asymptotic, we need to study the contributions coming from the two corners $\mathfrak{B}_{01} \cap \mathfrak{B}_{11}$ and $\mathfrak{B}_{10} \cap \mathfrak{B}_{11}$.

If we were not renormalizing the integral, the expansion would follow directly from the push-forward theorem of [26]. Furthermore, as pointed out in [20, pg. 128], if we were renormalizing using ρ_{10} instead of x, the same theorem could be applied. As it is, our situation is simple enough that we can proceed by direct computation (cf. example 3.2 in [19]).

By judicious choice of coordinate patches we can consider the two corners $\mathfrak{B}_{01} \cap \mathfrak{B}_{11}$ and $\mathfrak{B}_{10} \cap \mathfrak{B}_{11}$ separately. First, near $\mathfrak{B}_{01} \cap \mathfrak{B}_{11}$, we can choose the boundary defining functions to be given by

$$\rho_{11} = x, \quad \rho_{01} = \frac{\sqrt{t}}{x}.$$

The reason this corner will contribute log-terms to the expansion is heuristically explained by looking at the level sets of x near $\mathfrak{B}_{01} \cap \mathfrak{B}_{11}$ in Figure 2, and is unrelated to renormalization. To compute the contribution coming

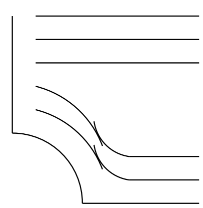

FIGURE 2. Lines we are integrating over

from the corner, we restrict our attention to the small region $\mathcal{U}_\epsilon$ defined by

$$\mathcal{U}_\epsilon := \{p \in \mathrm{diag}_H \mid 0 \leq \rho_{11}(p) \leq \epsilon, \ 0 \leq \rho_{01}(p) \leq \epsilon\}.$$

Thus, in the region $\mathcal{U}_\epsilon$, we have $\frac{\sqrt{t}}{\epsilon} \leq x \leq \epsilon$. Our choices of boundary defining functions give us a natural identification

$$\mathcal{U}_\epsilon = \partial M \times [0,\epsilon]_{\rho_{11}} \times [0,\epsilon]_{\rho_{01}}$$

and a natural projection $\pi_{\partial M} : \mathcal{U}_\epsilon \to \partial M$ onto the left factor. From (A.1), we have that in the region $\mathcal{U}_\epsilon$,

$$\begin{aligned}\mathcal{K}|_{\mathcal{U}_\epsilon} \in \rho_{01}^{-n+1}\rho_{11}^{-h}\mathcal{C}^\infty(\mathcal{U}_\epsilon; \pi_{\partial M}^*\Omega(\partial M))d\rho_{11}d\rho_{01} \\ = \rho_{01}^{-n}x^{-h-2}\mathcal{C}^\infty(\mathcal{U}_\epsilon; \pi_{\partial M}^*\Omega(\partial M))dxdt.\end{aligned}$$

We therefore have the following asymptotic expansion at the corner $\mathfrak{B}_{01} \cap \mathfrak{B}_{11}$,

$$\begin{aligned}\mathcal{K}|_{\mathcal{U}_\epsilon} &\sim \sum_{k=-n}^{\infty} \sum_{\ell=-h-2}^{\infty} a_{k\ell}\rho_{01}^k x^\ell \, dx \, dt, \quad a_{k\ell} \in \mathcal{C}^\infty(\partial M; \pi_{\partial M}^* \Omega(\partial M)), \\ &\sim \sum_{k=-n}^{\infty} \sum_{\ell=-h-2}^{\infty} a_{k\ell}\sqrt{t}^k x^{\ell-k} \, dx \, dt.\end{aligned} \tag{A.4}$$

If we write $\mathcal{K}|_{\mathcal{U}_\epsilon} = a dx dt$ for some appropriate section a of $\pi_{\partial M}^*\Omega(\partial M)$, then the integral of $\mathcal{K}|_{\mathcal{U}_\epsilon}$ over the slice $\sqrt{t} = C$ can be written

$$\int_{\mathcal{U}_\epsilon \cap \{t=C^2\}} \mathcal{K} = \left(\int_{\frac{C}{\epsilon}}^{\epsilon} \left(\int_{\partial M} a \right) dx \right) dt$$

In particular, for the terms in the asymptotic expansion (A.4), we get for $\ell - k \neq -1$

$$\left(\int_{\frac{C}{\epsilon}}^{\epsilon} \left(\int_{\partial M} a_{k\ell} \right) C^k x^{\ell-k} dx \right) dt = \\ \left(\int_{\partial M} a_{k\ell} \right) \frac{1}{\ell - k + 1} \left(C^k \epsilon^{\ell-k+1} - \frac{C^{\ell+1}}{\epsilon^{\ell-k+1}} \right) dt,$$

while for $\ell - k = -1$,

$$\left(\int_{\frac{C}{\epsilon}}^{\epsilon} \left(\int_{\partial M} a_{k\ell} \right) C^k x^{\ell-k} dx \right) dt = -\left(\int_{\partial M} a_{k\ell} \right) C^k \log\left(\frac{C}{\epsilon^2} \right).$$

Thus,

$$\int_{\mathcal{U}_\epsilon \cap \{t=C^2\}} \mathcal{K}|_{\mathcal{U}_\epsilon} \sim C^{-n} \sum_{k=0}^{\infty} \alpha_k C^k + C^{-(h+1)} \sum_{k=0}^{\infty} \beta_k C^k \log C$$

when $C \searrow 0$, which gives an asymptotic expansion of the form given in (A.3).

Next, near $\mathfrak{B}_{10} \cap \mathfrak{B}_{11}$, we can choose the boundary defining functions to be given by

$$\rho_{11} = \sqrt{t}, \quad \rho_{10} = \frac{x}{\sqrt{t}}$$

Heuristically, from Figure 2, one would not expect the corner $\mathfrak{B}_{01} \cap \mathfrak{B}_{11}$ to contribute log-terms to the expansion, however we shall see that the renormalization of the integrals causes these log-terms to appear. We can consider the neighborhood

$$\mathcal{V}_\epsilon = \{p \in \operatorname{diag}_H \mid 0 \leq \rho_{11}(p), \rho_{10}(p) \leq \epsilon\}$$

of $\mathfrak{B}_{10} \cap \mathfrak{B}_{11}$ in diag_H. Again, we have an identification $\mathcal{V}_\epsilon = \partial M \times [0,1]_{\rho_{11}} \times [0,1]_{\rho_{10}}$ and a natural projection $\pi_{\partial M} : \mathcal{V}_\epsilon \to \partial M$ onto the left factor.

According to (A.1), we have

$$x^z\mathcal{K}|_{\mathcal{V}_\epsilon} \in \rho_{10}^{-1+z}\rho_{11}^{-h-1+z}\mathcal{C}^\infty(\mathcal{V}_\epsilon;\pi_{\partial M}^*\Omega(\partial M))\, d\rho_{10}\, dt$$

with corresponding asymptotic behavior

$$\begin{aligned} x^z\mathcal{K}|_{\mathcal{V}_\epsilon} &\sim \sum_{k=-1}^{\infty}\sum_{\ell=-h-1}^{\infty} \widetilde{a}_{k\ell}\rho_{10}^{k+z}\rho_{11}^{\ell+z}\, d\rho_{10}\, dt, \quad \widetilde{a}_{k\ell}\in\mathcal{C}^\infty(\partial M;\Omega(\partial M)),\\ &\sim \sum_{k=-1}^{\infty}\sum_{\ell=-h-1}^{\infty} \widetilde{a}_{k\ell}x^{k+z}(\sqrt{t})^{\ell-k-1}\, dx\, dt. \end{aligned}$$

For each k and ℓ and for $\operatorname{Re} z \gg |k|, |\ell|$, one computes that $\widetilde{a}_{k\ell}$ contributes to the asymptotic expansion of $\int_{\mathcal{V}_\epsilon\cap\{t=C^2\}} x^z\mathcal{K}|_{\mathcal{V}_\epsilon}$ via

$$\left(\int_0^{C\epsilon}\left(\int_{\partial M}\widetilde{a}_{k\ell}\right)C^{\ell-k-1}x^{k+z}\, dx\right)\, dt = \left(\int_{\partial M}\widetilde{a}_{k\ell}\right)\epsilon^{k+z+1}\frac{C^{\ell+z}}{k+z+1}\, dt.$$

Finally taking the finite part at $z=0$, we get a contribution of

$$\begin{cases} \left(\int_{\partial M}\widetilde{a}_{k\ell}\right)\epsilon^{k+1}\frac{C^\ell}{k+1}\, dt & \text{if } k\neq -1\\ \left(\int_{\partial M}\widetilde{a}_{k\ell}\right)C^\ell\left[\log\epsilon+\log C\right]\, dt & \text{if } k=-1 \end{cases}$$

since $(C\epsilon)^z = e^{z\log(C\epsilon)} = 1 + z\log(C\epsilon) + \mathcal{O}(z^2)$. Thus we see that

$$\int_{\mathcal{V}_\epsilon\cap\{t=C^2\}} x^z\mathcal{K}|_{\mathcal{V}_\epsilon} \sim C^{-(h+1)}\left(\sum_{k=0}^{\infty}\widetilde{\alpha}_k C^k + \sum_{k=0}^{\infty}\widetilde{\beta}_k C^k\log C\right),$$

which gives again an asymptotic expansion of the form (A.3) and completes the proof. □

Remark A.2. *As on a closed manifold, one can show that the expansion at* $\mathfrak{B}_{01}$ *of* $t^{n/2}$ *times the heat kernel involves only powers of* t *(instead of* $\sqrt{t}$*).*

Remark A.3. *As mentioned in the proof of the theorem, for integrable densitites the pushforward theorem gives the form of the expansion* (A.3) *with log-terms arising from the expansion at* $\mathfrak{B}_{01}\cap\mathfrak{B}_{11}$ *but not from the corner* $\mathfrak{B}_{10}\cap\mathfrak{B}_{11}$*. As an example, for a hyperbolic surface with cusps (*$n=2$*,* $h=0$*), we have an expansion*

$$\text{(A.5)} \qquad {}^R\operatorname{Tr}(e^{-t\Delta_\Sigma}) \sim \left[t^{-1}\sum_{k\geq 0}a_k t^{k/2} + t^{-1/2}\sum_{k\geq 0} b_k t^{\frac{k}{2}}\log t\right]\, dt$$

for the heat kernel of the Laplacian. From remark A.2, we know that the corner $\mathfrak{B}_{01}\cap\mathfrak{B}_{11}$ *leads to no logarithmic term and an explicit computation at the corner* $\mathfrak{B}_{10}\cap\mathfrak{B}_{11}$ *shows that* $b_0\neq 0$ *and is in fact the same as the corresponding term in the short time expansion of the "relative trace" considered by Müller* [31, equation (2.3)].

Corollary A.4. *For the operator D_{E_ρ} considered in* (5.4), *we have the asymptotic expansion*

$$\operatorname{Tr}(e^{-tD^2_{E_\rho}}) \sim \frac{1}{t}\sum_{k=0}^{\infty} a_k t^{\frac{k}{2}} \quad \text{as } t \searrow 0,$$

for some constants a_k, $k \in \mathbb{N}_0$.

Proof. From theorem A.1, we have an asymptotic expansion of the form (A.5). To see that the coefficient b_k vanishes for all $k \in \mathbb{N}_0$, notice first that since the vertical operator of D_{E_ρ} is invertible by assumption, we do not pick up any logarithmic terms from the corner $\mathfrak{B}_{10} \cap \mathfrak{B}_{11}$ in light of (A.2). At the other corner $\mathfrak{B}_{01} \cap \mathfrak{B}_{11}$ the asymptotic expansion of $\mathcal{K}|_{\mathrm{diag}_H}$ is dictated by the standard local expansion in the interior,

$$\mathcal{K}|_{\mathrm{diag}_H} \sim \frac{1}{t}\sum_{k=0}^{\infty} c_k t^k \, dr\, d\theta dt \tag{A.6}$$

in the coordinates of (2.6). Although a priori the coefficient c_k could depend on r and θ, it is in fact constant since it is a universal expression in terms of the curvature of E_ρ, which is zero, and the curvature of g_Σ, which is constant. Since we would need a term of the form $t^k r^{-1} dr d\theta dt$ to pick up a logarithmic term, we see that the asymptotic expansion of $\mathcal{K}|_{\mathrm{diag}_H}$ at the corner $\mathfrak{B}_{01} \cap \mathfrak{B}_{11}$ leads to no logarithmic term. Consequently, $b_k = 0$ for all $k \in \mathbb{N}_0$ and the result follows. □

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